PHYS 7810: Hydrodynamics **Spring 2024**

Homework 3

Due: March 5 at 11:59 PM. Submit on Canvas.

Problem 1 (Plasmons): In fluids with dynamical electromagnetism (which typically arise either in plasmas in space, or also in a fluid made out of mobile electrons in a metal), the nature of sound waves is qualitatively altered by Maxwell's equations. Let $\rho(x,t)$ denote the density of (charged) electrons. In thermodynamic equilibrium, $\rho(x,t) = \bar{\rho}$, and there is some compensating non-dynamical charge density $-\bar{\rho}$ such that the overall system is charge-neutral.

10 **A:** Let us begin by thinking about the thermodynamic ensemble, which we might approximate as

$$\Phi[\rho, \mathbf{E}] = \Phi_0[\rho] + \beta \int \mathrm{d}^3x \, \frac{\epsilon}{2}\mathbf{E}^2. \tag{1}$$

If the charged particles in the fluid move at non-relativistically slow speeds, we can neglect the contributions from magnetic fields. Use Maxwell's equations to argue that $\mathbf{E}$ and ρ are not independent; hence in Fourier space, on long wavelengths ($k \to 0$):

$$\Phi \approx \frac{\beta e^2}{2\epsilon} \int \mathrm{d}^3k \, \frac{|\rho(\mathbf{k})|^2}{k^2} + \Phi_0. \tag{2}$$

20 **B:** Using the modified Φ appropriate for this charged fluid (where $\rho \neq 0$ in thermodynamic equilibrium),[1] we will now calculate the quasinormal modes. For simplicity, you should only keep track of the leading order behavior in the real/imaginary parts of $\omega(k)$ in the long wavelength limit $k \to 0$.

- B1. Show that the viscous diffusive mode is unaffected by the modified Φ.
- B2. Show that the sound mode is qualitatively altered, and describe its new dispersion relation. Do not assume that the system has Galilean symmetry. This mode is called a **plasmon**.
- B3. Describe what happens to the incoherent diffusion mode.

5 **C:** Is this fluid more or less likely to behave as an incompressible fluid than a conventional charge-neutral fluid (like liquid water)? Give a short qualitative answer.

Problem 2 (Width of a shock wave): Consider a Galilean-invariant fluid (Lecture 9) in one spatial dimension ($d = 1$) with pressure (in a frame where the fluid is at rest) is given by

$$P(\mu_{\mathrm{th}}, T) = \frac{1}{2}\chi\mu_{\mathrm{th}}^2 + \frac{1}{2}cT^2. \tag{3}$$

10 **A:** Let us first sort out the thermodynamic properties of this fluid, following Lecture 8. Calculate the mass density, energy density, and entropy density of the fluid as a function of μ_{th} and T.

[1] *Hint:* The dissipative MSR Lagrangian from Lecture 8 is a useful starting point for this problem.

15 **B:** Consider a Riemann problem where the initial mass density $\rho_\mathrm{L} = 2\rho_\mathrm{R}$, and the initial velocity $v_\mathrm{R} = 0$ while $v_\mathrm{L} > 0$. Our goal is to look for energy densities $\epsilon_{\mathrm{L,R}}$ such that the solution to the Riemann problem consists of a single shock wave.

B1. Following Lecture 11, move to a reference frame in which the shock is stationary. Obtain a set of jump conditions relating the fluid parameters on the left and right of the shock wave.

B2. Now use the parameters above. Show that a single shock wave exists only when

$$\epsilon_\mathrm{L} = 5\epsilon_\mathrm{R}. \tag{4}$$

15 **C:** To confirm that this is a *physical* solution to the Riemann equation, we need to check that entropy is produced across the shock wave.

C1. Following Lecture 8, confirm using thermodynamic identities that in ideal hydrodynamics,

$$\partial_t s + \partial_x (sv) = 0. \tag{5}$$

C2. The equation above need not hold directly at the location of the shock wave. Show that the equation above only makes sense in the presence of a shock wave if (5) is modified to

$$\partial_t s + \partial_x (sv) = \alpha \cdot \delta(x - v_\mathrm{sh} t). \tag{6}$$

Assuming that $\chi \rho_\mathrm{L}^2 = cT_\mathrm{L}^2$, evaluate α for the Riemann problem above. Show that $\alpha > 0$.

10 **D:** The presence of bulk viscosity in the Navier-Stokes equations changes on "short enough" length scales the physics of the shock wave. Assume that this is the only non-zero dissipative coefficient.

D1. In Lecture 8, we argued that dissipation leads to entropy production in the Navier-Stokes equations. Without re-doing the derivation from scratch, argue that the entropy production rate in the presence of bulk viscosity will be

$$\partial_t s + \partial_x (\cdots) = \frac{\zeta}{T}(\partial_x v)^2. \tag{7}$$

D2. By equating the two formulas you have obtained for entropy production, *estimate* the width ℓ of the shock wave within viscous hydrodynamics, neglecting the precise O(1) prefactor. You do not need to explicitly solve the Navier-Stokes equations; a simple argument is enough.

Problem 3 (Vortex knots): In this problem, we will study the dynamics of vortices in an ideal and incompressible fluid, following Lecture 12. Recall that we derived the equation:

$$\partial_t \mathbf{r}(s) = \frac{\Gamma}{4\pi} \int \mathrm{d}s' \frac{\partial_s \mathbf{r}(s') \times (\mathbf{r}(s) - \mathbf{r}(s'))}{|\mathbf{r}(s) - \mathbf{r}(s')|^3}. \tag{8}$$

15 **A:** In general, (8) is a very complicated integral equation to solve. However, under certain approximations, we might find reasonable approximate solutions.

A1. Suppose that we set

$$\mathbf{r}(s') \approx \mathbf{r}(s) + (\partial_s \mathbf{r}(s))(s' - s) + \frac{1}{2}\left(\partial_s^2 \mathbf{r}(s)\right)(s' - s)^2 + \cdots. \tag{9}$$

Show that we may then approximate

$$\partial_t \mathbf{r} = C \partial_s \mathbf{r} \times \partial_s^2 \mathbf{r}, \tag{10}$$

where the constant C arises out of a log-divergent integral; in what follows, take it to be constant.

A2. Let us parameterize the point $\mathbf{r}$ in cylindrical coordinates (r, ϕ, z). Show that

$$C^{-1}\partial_t r = r\partial_s\phi\partial_s^2 z - 2\partial_s r\partial_s\phi\partial_s z - r\partial_s^2\phi\partial_s z, \tag{11a}$$

$$C^{-1}\partial_t \phi = \frac{\partial_s^2 r\partial_s z - \partial_s r\partial_s^2 z}{r} - (\partial_s\phi)^2\partial_s z, \tag{11b}$$

$$C^{-1}\partial_t z = 2(\partial_s r)^2\partial_s\phi + r\partial_s r\partial_s^2\phi + r^2(\partial_s\phi)^3 - r\partial_s^2 r\partial_s\phi. \tag{11c}$$

A3. Verify that the following **vortex ring** is a solution:

$$r = R, \tag{12a}$$

$$\phi = \frac{s}{R}, \tag{12b}$$

$$z = \frac{Ct}{R}. \tag{12c}$$

15 **B:** Let $\mathbf{r}_0$ denote the solution found in (12).

B1. Consider now the linear stability of this solution, by setting

$$\mathbf{r}(s,t) = \mathbf{r}_0(s,t) + \mathbf{r}_1(s,t). \tag{13}$$

Solve (11) for $(r_1, \phi_1, z_1) \sim \mathrm{e}^{\mathrm{i}(ks-\omega t)}$, treating $\mathbf{r}_1$ as an infinitesimal perturbation and thus only keeping first order corrections in $\mathbf{r}_1$ after plugging in to (11) . Deduce that

$$\omega = \pm Ck\sqrt{k^2 - \frac{1}{R^2}}. \tag{14}$$

B2. Explain why there must be integers p and q for which

$$k = \frac{q}{pR}. \tag{15}$$

B3. What do you find happens when $p = 1$? As part of your answer, draw physically what the vortex configurations will look like.

B4. Argue that when $p > 1$, the resulting vortex configuration is a **vortex knot** that appears to wrap a torus (donut). Discuss, and explain intuitively, the stability criteria for these vortex knots.