PHYS 7810: Hydrodynamics **Spring 2024**

Homework 4

Due: March 19 at 11:59 PM. Submit on Canvas.

Problem 1 (Unstable flow between rotating cylinders): Consider a fluid flowing between two cylinders of radius $R_1 < R_2$, with the smaller cylinder centered in the larger one. In polar coordinates, the fluid exists for $R_1 < r < R_2$. Assuming, as we will for this problem, that the flow has rotational symmetry ($\partial_\theta = 0$), the incompressible Navier-Stokes equations read:

$$\frac{1}{r}\partial_r \left(r v_r\right) + \partial_z v_z = 0, \tag{1a}$$

$$\partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r - \frac{v_\theta^2}{r} + \frac{\partial_r P}{\rho} = \nu\left(\nabla^2 v_r - \frac{v_r}{r^2}\right), \tag{1b}$$

$$\partial_t v_\theta + v_r \partial_r v_\theta + v_z \partial_z v_\theta + \frac{v_\theta v_r}{r} = \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right), \tag{1c}$$

$$\partial_t v_z + v_r \partial_r v_z + v_z \partial_z v_z + \frac{\partial_z P}{\rho} = \nu \nabla^2 v_z. \tag{1d}$$

In this problem, we will consider the flow subject to no-slip boundary conditions, in the presence of two rotating cylinders. Assume that the inner cylinder rotates with angular frequency Ω_1, and the outer cylinder with Ω_2. The no-slip boundary condition thus enforces $v_\theta(r = R_{1,2}) = R_{1,2}\Omega_{1,2}$.

15 **A:** Show that a simple solution to the Navier-Stokes equations, obeying the correct boundary conditions, has $v_r = v_z = 0$ while

$$v_\theta(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}\frac{1}{r}. \tag{2}$$

25 **B:** Now consider the stability of this flow pattern.

- B1. Let $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ and $P = P_0 + P_1$. Expand the Navier-Stokes equations to first order in the perturbations.
- B2. Now, let us try to reduce these equations to a single ordinary differential equation for, e.g.,

$$v_{\theta 1} = g(r)\mathrm{e}^{\mathrm{i}kz - \mathrm{i}\omega t}. \tag{3}$$

Show that in the limit $W = R_2 - R_1 \ll R_1$ and $|\Omega_1 - \Omega_2| \ll |\Omega_1|$, you may approximate that

$$\left(\mathrm{i}\omega - \nu k^2 + \nu \partial_r^2\right)^2 \left(k^2 - \partial_r^2\right) g(r) = 2\Omega_1 \left(2\Omega_1 + \frac{R_1}{W}(\Omega_2 - \Omega_1)\right) k^2 g(r). \tag{4}$$

- B3. Show that if you use the rather unusual boundary conditions that $v_{r1} = v_{\theta 1} = 0$ but $\tau_{rz} = 0$ at $r = R_{1,2}$, then all six boundary conditions are solved by the simple ansatz

$$g(r) = \sin\frac{n\pi(r - R_1)}{W}, \quad (n = 1, 2, 3, \ldots). \tag{5}$$

B4. Show that the system is unstable so long as

$$\nu \leq \nu_{\mathrm{c}} = \frac{2W^2\Omega_1}{\pi^2}\sqrt{\frac{2}{27}\left(2 + \frac{R_1}{W}(\Omega_2 - \Omega_1)\right)}. \tag{6}$$

Although the boundary conditions we have chosen are unrealistic, choosing more realistic boundary conditions doesn't lead to a qualitatively different answer, while significantly complicating the calculation. Historically, an accurate determination of the onset of instability in such a flow was important in experimentally deducing that no-slip boundary conditions were the physically correct boundary conditions.

Problem 2 (Blood flow): In this problem we will study the problem of blood flow through the circulatory system. Consider a blood vessel as a cylinder of radius R, oriented in the z-direction. Suppose that there is a pressure[1]

$$P = -\alpha z\cos(\omega t) = -\alpha z \times \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\omega t}\right] \tag{7}$$

which drives the flow of blood, which has kinematic viscosity ν and mass density ρ. Treat blood as incompressible.

20 **A:** Let us first find the flow pattern as a function of time.

A1. Show that a consistent solution can be found with only $v_z \neq 0$.

A2. Show that the velocity is given by

$$v_z = \mathrm{Re}\left[-\frac{\alpha \mathrm{e}^{-\mathrm{i}\omega t}}{\mathrm{i}\omega\rho}\left(1 - \frac{\mathrm{J}_0(\sqrt{\mathrm{i}\omega/\nu}r)}{\mathrm{J}_0(\sqrt{\mathrm{i}\omega/\nu}R)}\right)\right]. \tag{8}$$

Here J_0 is the Bessel function of order 0.

A3. Sketch the flow profile when ω is small, and when it is large. What does ω need to be small/large compared to?

15 **B:** How much power must the human heart supply to push blood through the human body?

B1. Show that the time-averaged power needed to drive the flow through a pipe of length L is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\pi\alpha^2 L}{\rho\omega}\int_0^R \mathrm{d}r\, r\,\mathrm{Im}\left(\frac{\mathrm{J}_0(\sqrt{\mathrm{i}\omega/\nu}r)}{\mathrm{J}_0(\sqrt{\mathrm{i}\omega/\nu}R)}\right). \tag{9}$$

B2. In the limit $\omega \to 0$, evaluate this expression analytically, and compare to the results from Lecture 12, where an analogy was made between resistor networks and pipe networks. Does your expression for power make sense in this context?

B3. The largest blood vessel in the body has radius $R \sim 2$ cm and $L \sim 0.5$ m; the pressure gradient across it might be $\alpha \sim 3 \times 10^3$ Pa/m, while the frequency $\omega \sim 0.2$ s^{-1}. Given that blood has $\rho \sim 10^3$ kg/m^3 and $\nu \sim 10^{-5}$ m^2/s, estimate the power needed to drive the flow pattern above. You may want to evaluate the integral above numerically in `Mathematica`. Compare your answer to the experimentally-deduced value of ~ 1 W.

[1] In this problem, it will help to use complex-valued variables, although the physical response must of course be real-valued.

Problem 3 (Jets): A jet corresponds to a fluid flow formed by the injection of a constant momentum per unit time F (i.e. a force) applied at the point $(x, y) = (0, 0)$. We will study the evolution of such a jet as it expands in a two-dimensional flow.

In this problem, you should assume that the fluid is incompressible, with density ρ. Define the parameter

$$J = \frac{F}{\rho}, \tag{10}$$

and assume the fluid has dynamical viscosity ν.

25 **A:** We first begin by solving the problem of a laminar jet. We will invoke the boundary layer theory of Lecture 14 as an approximation for the evolution of the jet.

- A1. Following Lecture 14, non-dimensionalize the stream function ψ and coordinates x and y, given the two dimensional parameters J and ν in the problem.
- A2. By demanding that

$$\int\limits_{-\infty}^{\infty} \mathrm{d}y \; v_x^2 = J, \tag{11}$$

deduce that a similarity solution for the stream function will take the dimensionless form

$$\Psi = X^{1/3} f\left(\frac{Y}{X^{2/3}}\right). \tag{12}$$

- A3. What ordinary differential equation must f obey?
- A4. Show that a solution satisfying your result, along with (11) and sensible boundary conditions, is

$$f(\xi) = \left(\frac{9}{2}\right)^{1/3} \tanh \frac{\xi}{48^{1/3}}. \tag{13}$$

Note that the constant c is dimensionless (and you should deduce what its value needs to be), but might depend on the precise non-dimensionalization you found earlier.

- A5. Show that the jet **entrains** nearby fluid into it, by calculating the particle flow rate

$$Q(x) = \int\limits_{-\infty}^{\infty} \mathrm{d}y \; v_x. \tag{14}$$

Determine how rapidly $Q(x)$ increases at large x.

- A6. Argue that our model of a jet is only accurate for $x \gtrsim x_\mathrm{c}$. What is the value of x_c and why?

15 **B:** Now let us consider the theory of a turbulent jet. As in Lecture 16, you may use a theory of weak turbulence to estimate the flow, by incorporating a turbulent viscosity into the theory.

- B1. First, attempt to generalize the similarity solution for the jet's flow pattern. By dimensional analysis, write down the most general possible stream function ψ.
- B2. Use boundary layer theory, with the appropriate turbulent viscosity[2] and your ansatz from B1, to predict the shape of the turbulent wake.
- B3. Does the turbulent wake entrain more or less of the ambient fluid than the laminar wake?

[2] *Hint:* Take ν_t to depend on x but not ψ for simplicity.