PHYS 7810: Hydrodynamics **Spring 2024**

Homework 5

Due: April 16 at 11:59 PM. Submit on Canvas.

Problem 1 (Rectangular fluid): Consider a two-dimensional anisotropic and incompressible fluid. We presume that the equations of motion take the form (here ρ is constant):

$$\partial_i v_i = 0, \tag{1a}$$

$$\rho v_j \partial_j v_i + \partial_i P = \eta_{jikl} \partial_j \partial_k v_l. \tag{1b}$$

Suppose that the only spacetime symmetries are separate mirror symmetries under $x \to -x$ and $y \to -y$. Do *not* assume that the fluid has microscopic time-reversal symmetry.

15 **A:** Let us first analyze the allowed form of η_{jikl}.

- A1. Suppose that the bulk viscosity $\delta_{ij}\delta_{kl}\eta_{jikl} = 0$. Show that there remain 5 independent coefficients allowed in η_{ijkl}, allowed by the spatial symmetries.
- A2. What are the constraints on these coefficients that arise from our dissipative hydrodynamic effective field theory, if any?

25 **B:** Suppose that we want to experimentally measure all of the independent viscosity coefficients in this system. Suppose for simplicity that the only experiments you can perform are to etch long rectangular channels (of width w) in the fluid. The channel need not be aligned with the x/y-symmetry axes of the fluid. Down the channel, you may apply a uniform pressure gradient of known magnitude; you may then measure the pressure gradient *across the width* of the channel, and you can also measure the flow rate of the fluid down the channel.

- B1. Let θ denote the angle made by the channel relative to the fluid, and consider the new coordinates:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

  Argue that the channel flow solution will have only $v_X \neq 0$, and that $v_X(Y)$ is only a function of a single coordinate.

- B2. Assuming no-slip boundary conditions, calculate the flow $v_X(Y)$ for a given pressure gradient $\partial_X P$, and deduce the total flow rate

$$Q = \int_0^w \mathrm{d}Y \; \rho v_X(Y). \tag{3}$$

- B3. Calculate the pressure difference between the two boundaries of the channel.
- B4. By rotating the channel (varying θ), is it possible to deduce all of the independent viscosities of the rectangular fluid? Why or why not?

B5. If time-reversal symmetry is restored, is it generically possible to see a “Hall” pressure gradient across the width of the channel?

5 **C:** In a system with rectangular symmetry, and momentum and particle number conservation, do you expect that the hydrodynamic equations would take the form of (1)? Why or why not?

Problem 2 (Bubble oscillations): Consider a spherical bubble of radius R, containing fluid of density ρ and pressure P. The outer surface of the bubble has surface tension with coefficient α, separating it from a vacuum. There are no external forces acting on the bubble. As in Lecture 17, we can model the energy stored in surface tension as

$$U = \alpha \int \mathrm{d}\theta \mathrm{d}\phi \sqrt{(R+\zeta)^4 \sin^2\theta + (R+\zeta)^2 \sin^2\theta (\partial_\theta \zeta)^2 + (R+\zeta)^2 (\partial_\phi \zeta)^2}. \tag{4}$$

15 **A:** Suppose that the radius of the bubble is deformed to $r = R + \zeta$ for $\zeta \ll R$.

A1. As in Lecture 17, we expect that the radial pressure drop across the bubble obeys

$$P(r > R) - P(r < R) = -\frac{1}{(R+\zeta)^2 \sin\theta} \frac{\delta U}{\delta \zeta}. \tag{5}$$

Deduce that even in equilibrium ($\zeta = 0$) there must be a relative pressure drop between the fluid outside vs. inside the bubble. Is the pressure larger inside or outside?

A2. Suppose that the inside of the bubble is filled with an incompressible fluid that undergoes irrotational flow. If the velocity potential is Φ, deduce that one boundary condition on Φ is

$$\frac{\partial \Phi(r=R)}{\partial t} = -\frac{2\alpha}{\rho R^2}\zeta - \frac{\alpha}{\rho R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \zeta}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\zeta}{\partial\phi^2}\right]. \tag{6}$$

A3. What is the other boundary condition linking Φ and ζ?

15 **B:** Now solve for the normal modes of the bubble.

B1. What equation does Φ obey for $r < R$? Deduce the most general possible solution for Φ.

B2. Use the boundary conditions from A2 and A3 to deduce the frequencies of the normal modes.

B3. Sketch the shape of the bubble in the lowest frequency normal mode with $\omega \neq 0$. Give a physical and intuitive explanation for why this is the slowest normal mode of the bubble.

Problem 3 (Atmospheric waves): In this problem we will consider the propagation of very long wavelength gravity waves in the atmosphere. As discussed in Lecture 18, atmospheric fluid dynamics is best considered in a non-inertial reference frame rotating with the Earth, and so the Navier-Stokes equations need to be modified to account for Coriolis and centrifugal forces. As in Lecture 17, you should neglect viscous corrections in the analysis of gravity waves; further neglect surface tension for this problem.

25 **A:** Let us begin by studying the following crude model of an atmosphere, as a fluid of height h in equilibrium. The effective acceleration due to gravity will be g (this includes corrections from centrifugal forces, as discussed in Lecture 18), while the Coriolis force is proportional to the parameter f. Unlike in Lecture 17, due to the Coriolis force, we can no longer look for irrotational gravity waves. As in Lecture 17, the incompressible fluid exists for $0 \leq z \leq h + \zeta(x,y)$, with $|\zeta| \ll h$.

A1. Write down the equations of motion for pressure and the 3 components of velocity. Keep only first order perturbations away from an equilibrium configuration where the fluid is at rest. Explain what the most general solution of these equations is, up to boundary conditions.

A2. Impose appropriate boundary conditions at $z = 0$ and $z = h + \zeta$.

A3. Look for plane wave solutions proportional to, e.g., $v_x \sim v_x(z)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}$. Find the dispersion relation $\omega(k)$ for all possible modes.

A4. Show that if $k \to 0$, you find three possible solutions: one with $\omega = 0$ and the other with

$$\omega = \pm\sqrt{f^2 + ghk^2}. \tag{7}$$

Explain the physical origin of all three of these normal modes.

20 **B:** Now let us discuss the propagation of these atmospheric waves near the equator.

B1. Show that in the long wavelength limit, there is a systematic way to "integrate out" the z direction and obtain coupled equations for ζ and $v_{x,y}$ as functions of x, y, t:

$$0 = \partial_t\zeta + h\left(\partial_x v_x + \partial_y v_y\right), \tag{8a}$$

$$0 = \partial_t v_x + g\partial_x\zeta - f v_y, \tag{8b}$$

$$0 = \partial_t v_y + g\partial_y\zeta + f v_x. \tag{8c}$$

B2. Recall from Lecture 18 that if θ_L denotes the latitude of the Earth, that $f \sim \sin\theta_\mathrm{L}$. Hence crossing the equator ($\theta_\mathrm{L} = 0$), the sign of f flips. As a cartoon model of this effect, take the model (8) along with

$$f(y) = \mathrm{sign}(y) \cdot f_0 \tag{9}$$

where $f_0 > 0$ is a constant. Find the normal modes of this model, and discuss what you find. Think carefully about what happens near the interface at $y = 0$, and about what the physically correct boundary conditions must be.