Homework 6

Due: April 23 at 11:59 PM. Submit on Canvas.

Problem 1 (Viscosity of air): In Lecture 23 we used the relaxation time approximation to argue that the viscosity of a weakly-interacting classical gas was $\eta = P\tau$. To compare with actual experimental data for a $d = 3$ dimensional gas, we need to actually calculate τ. This requires unpacking a bit more the rate $\mathcal{R}$ in the collision integral. To proceed, we quote some results from classical scattering theory:

$$\mathcal{R}\left(\mathbf{p}_1\mathbf{p}_2 \to \mathbf{p}_1'\mathbf{p}_2'\right) = \int \mathrm{d}^2\mathbf{x}_\perp \cdot \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{m} \times \delta(\mathbf{p}_1' - \mathbf{p}_1'(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x}_\perp))\delta(\mathbf{p}_2' - \mathbf{p}_2'(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x}_\perp)) \tag{1}$$

where $\mathbf{x}_\perp$ corresponds to the "impact parameter" (i.e. the relative spatial offset of the incident particles far from the collision), and $\mathbf{p}_1'(\mathbf{p}, \mathbf{p}_2, \mathbf{x}_\perp)$ is the final momentum of particle 1 given the initial momenta and impact parameter. The overall prefactor $\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{m}\mathrm{d}^2\mathbf{x}_\perp$ counts the rate at which the particles actually approach each other, per particle per unit volume.

A: Let us approximate that the gas is made up of perfect spheres of radius a. It is helpful to separate out the center of mass momentum by defining

$$\mathbf{p}_1 = \overline{\mathbf{p}} + \mathbf{q}, \tag{2a}$$
$$\mathbf{p}_2 = \overline{\mathbf{p}} - \mathbf{q}. \tag{2b}$$

Note that if $\mathbf{q} = mu\hat{\mathbf{z}}$, then in the center of mass frame as $t \to -\infty$,

$$\mathbf{x}_1(t) = (x_\perp, y_\perp, ut), \tag{3a}$$
$$\mathbf{x}_2(t) = (0, 0, -ut). \tag{3b}$$

If $\mathbf{p}_1 + \mathbf{p}_1'$ is parallel to the tangent plane between the surface of the spheres at the moment of collision (i.e. they reflect off each other), deduce $\mathbf{p}_1'$ and $\mathbf{p}_2'$.

B: With a specific model for collisions, let us now predict viscosity.

B1. Following the variational method discussed in Lecture 24, estimate the shear viscosity given the collision integral of **A** by using the trial $|\psi\rangle = |p_x p_y\rangle$. Show that in this approximation,[1]

$$\eta \approx \frac{5\sqrt{mT}}{64\sqrt{\pi}a^2}. \tag{4}$$

B2. Predict the viscosity of air at room temperature ($T \approx 300$ K or $T \approx 4 \times 10^{-21}$ J). Assume that air is made up of density $n \approx 10^{27}$ m^{-3} of N_2 molecules, each of which has molecular mass $m \approx 5 \times 10^{-26}$ kg, and size $a \approx 3 \times 10^{-10}$ m. Compare to the experimental value, which was given (as kinematic viscosity) in class.

[1] *Hint:* The best way to proceed is to evaluate the linearized collision integral *carefully* in the coordinate system (2). Use rotational symmetry to show that $10\langle p_x p_y|\mathsf{W}|p_x p_y\rangle = \langle p_i p_j - \frac{1}{3}p^2\delta_{ij}|\mathsf{W}|p_i p_j - \frac{1}{3}p^2\delta_{ij}\rangle$.

Problem 2 (Anisotropic gas): Consider the kinetic theory of an anisotropic gas in which the kinetic energy (i.e. "dispersion relation") of the single-particle excitations takes the form

$$\epsilon(p_x, p_y) = \frac{a}{4}\left(p_x^4 + p_y^4\right). \tag{5}$$

Assume that the (microscopic) dynamics is time-reversal-symmetric, and that in equilibrium the distribution function takes the form

$$f_{\mathrm{eq}}(\mathbf{p}) = \mathrm{e}^{-\beta(\epsilon(\mathbf{p})-\mu)}. \tag{6}$$

Ignore energy conservation (you can imagine that our gas can exchange energy, but not momentum, with some other immobile degrees of freedom), but assume particle number and momentum are still conserved. This reduces the number of calculations you need to do.

20 **A:** Following Lecture 22, let us deduce the ideal hydrodynamics of this system. Assuming that

$$f(\mathbf{x}, \mathbf{p}) = \mathrm{e}^{-\beta(\epsilon(\mathbf{p})-\mu_{\mathrm{th}}(\mathbf{x})-\mathbf{v}(\mathbf{x})\cdot\mathbf{p})}, \tag{7}$$

calculate the number density n, number current J_i, momentum density g_i and stress tensor τ_{ji} in terms of the hydrodynamic variables μ_{th} and v_i (as well as the constant β). There is not a nice closed form expression for these functions to all orders in v_i; provide formulas valid up to $\mathrm{O}(v^4)$. Use `Mathematica` to evaluate integrals; you may want to define $C_n = 2^{(n-1)/2}(a\beta)^{-(n+1)/4}\Gamma(\frac{n+1}{4})$.

25 **B:** Now follow Lectures 23 and 24 to calculate dissipative corrections within linear response. Use the relaxation time approximation: on any "fast mode" in kinetic theory,

$$\mathsf{W}_{\mathrm{f}}|\Phi_{\mathrm{f}}\rangle = \frac{1}{\tau}|\Phi_{\mathrm{f}}\rangle. \tag{8}$$

Explicitly calculate the hydrodynamic constitutive relations to first order in derivatives – i.e. calculate the viscosity tensor and any other coefficients that may arise in the kinetic theory.

15 **C:** Let us now compare to effective field theory predictions, following Lecture 19.

C1. Are the thermodynamic constitutive relations from **A** the most general possible allowed by a time-reversal symmetric MSR Lagrangian?

C2. Within linear response around $v_i = 0$, what are all of the dissipative coefficients you would predict from the MSR Lagrangian? Are all of them non-zero in your calculation from **B**?

15 **Problem 3 (No subdiffusion in kinetic theory):** Consider a generic kinetic theory in which the linearized Boltzmann equation takes the form (following Lecture 23):

$$\mathrm{i}\omega|\Phi\rangle = \mathrm{i}k_i V_i|\Phi\rangle + \mathsf{W}|\Phi\rangle. \tag{9}$$

1. Give a physical reason why W must be positive semidefinite, regardless of any underlying symmetries in the problem.

2. Suppose that the combination of inversion and time-reversal (IT) is a symmetry. Fixing a specific orientation for the wave number $k_i = k \cdot n_i$ (with unit vector n_i), let

$$\omega(k) = \omega_0 + \omega_1 k + \omega_2 k^2 + \omega_3 k^3 + \cdots \tag{10}$$

be an eigenvalue of (9). Show that at least one of ω_0, ω_1, or ω_2 must be non-zero.

3. Does the conclusion hold if IT-symmetry is broken? If yes, show why; if no, find a counterexample.

This problem puts strong constraints on finding subdiffusive hydrodynamic universality classes (in which $\omega \sim k^n$ for $n > 2$) within the regime of validity of kinetic theory.