PHYS 7810
Hydrodynamics
Spring 2024

Lecture 1
Random walks

January 16

Office hours: 3-4 PM Fridays

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submit HW here

PHYS 7810, Spring 2024

- **Home**
- Assignments
- Grades
- People
- Pages
- Zoom

lectures are recording

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everything here

General Course Documents:

- Syllabus (course specific)
- Syllabus (university required statements)
- Holistic grading policy
- Course Slack (invitation link)

Lectures: notes/slides here

Lecture 1	January 16	Random walks
Lecture 2	January 18	Stochastic differential equations

Zoom

Homework:

Homework 1	due February 6	Solutions
Homework 2	due February 20	Solutions

No books/refs (now).

GRADES: 100% HW

↑ 6 (or 7)

Problem 1 (Microscopic model of a random walk): In Lectures 1 and 2 we discussed various cartoons for a random walk. These theories can be thought of as the effective theory for a more microscopic model, one of which we will consider in this problem: a "free particle" in one dimension which can exchange kinetic energy with a thermal bath. Consider a particle of mass m with position x and momentum p, with MSR Lagrangian (here γ and η are constants)

$$L = \pi\dot{x} + \sigma\dot{p} - \pi\frac{p}{m} + \sigma\left(\mathrm{i}\gamma\sigma + \eta p\right). \tag{1}$$

points

10 **A:** Give a clear physical interpretation of this theory.

A1. What would be the naive (noise-free) equations of motion? What is the noise?

A2. What is the Itō Fokker-Planck equation for this stochastic theory?

earn:

0, 2, 4, 6, 8, 10

(holistic grading)

HW graded out of 100

can earn ~115-125 points.

Extensions: 2 HW extensions ($\leq$ 48 hrs)

or drops. (auto-applied)

Hydrodynamics = dissipative effective field theory of thermalization in many-body system.

thermalization = dynamics that leads system to appear <u>locally</u> in a thermal state:

$$\langle x_i \rangle \approx \frac{\int dV\, e^{-\beta E} x_i}{\int dV\, e^{-\beta E}} \qquad \left[\langle x_i\rangle = \frac{\mathrm{tr}(e^{-\beta H} x_i)}{\mathrm{tr}(e^{-\beta H})}\right]$$

dissipative = thermalization is "irreversible" (2^{nd} law)

effective field theory = <u>systematic</u>, <u>most general</u> model consistent w/ symmetries, slow DOF

MANY examples of hydro:

liquids (water, oil...)
gases

ultracold atoms
quark-gluon plasma
electrons / phonons in solid-state

all described by some Navier-Stokes equations

Historically: this was hydro... ($\lesssim$ 1960)

(visco) elastic solids
superfluid

spontaneous symmetry breaking

diffusion of dye in water
spin fluct. in magnet

- Hydro EFT hold whether microscopic dynamics is classical or quantum
- Hydro EFT is always classical !

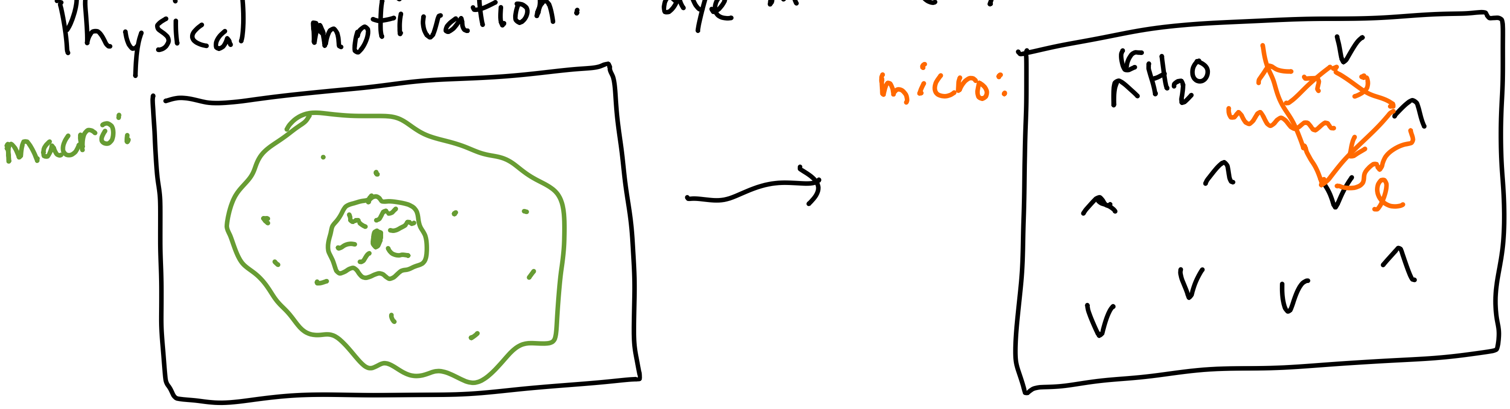
This class is about hydro as an EFT.
NOT center Navier-Stokes (engineering!)

Course outline:

① effective theory that's dissipative ? (Lagrangian ?)
② diffusion, simplest hydro
↓
③ Navier-Stokes
↓
④ hydro from kinetic theory
⑤ exotic symmetry, SSB, ...

Today: exactly solvable microscopic cartoon for diffusion:
discrete random walk.

Physical motivation: dye in water / diffusion.

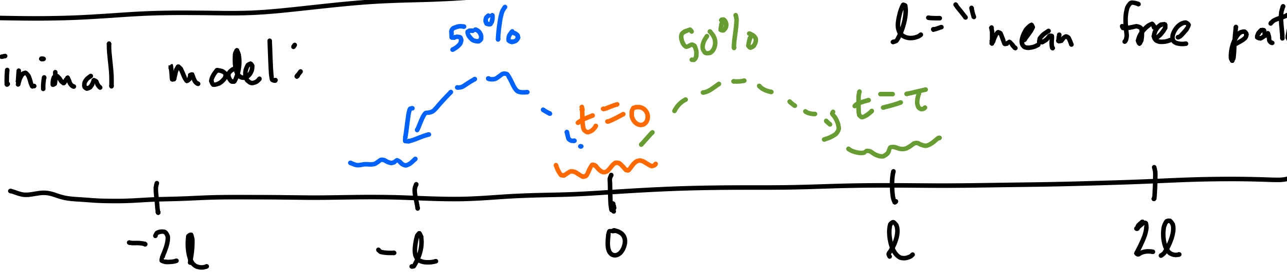


Where does dye molecule go as $t \to \infty$:
on average?

Better: probability distribution of molecules.

Math minimal model:

ℓ = "mean free path"



Probability $\mathbb{P}(\Delta x = \pm \ell) = 1/2$.

Goal: calculate $\mathbb{P}(x(n \cdot \tau) = m \cdot \ell) = ?$

Useful: work in "natural units" are $\ell = \tau = 1$.

Idea: introduce random variables...

• walker's position $x_n = x(n\tau)$

$$= \sum_{j=1}^{n} z_j$$

did we move left or right at step j?

$$z_j = \begin{cases} +1 & \text{moved right} \\ -1 & \text{left} \end{cases} \qquad \mathbb{P}(z_j = \pm 1) = 1/2.$$

Calculate $\mathbb{P}[x_n = m]$ using generating function method:

Define $G_n(y) = \sum_m y^m \mathbb{P}[x_n = m]$

Calculate $G_n(y)$ as expectation value $\mathbb{E}[\cdots]$, $\langle \cdots \rangle$:

$$\langle X \rangle = \sum_a a\, \mathbb{P}(X = a)$$

And: $G_n(y) = \langle y^{x_n} \rangle = \langle y^{\sum_{j=1}^n z_j} \rangle = \langle \prod_{j=1}^n y^{z_j} \rangle$.

(Implicit / physical) assumption: z_j's are independent:

$$\mathbb{P}(z_j = a \;\&\; z_{j'} = b) = \mathbb{P}(z_j = a)\,\mathbb{P}(z_{j'} = b).$$

And: $\langle y^{z_1} y^{z_2} \rangle = \sum_{a_1, a_2 = \pm 1} y^{a_1} y^{a_2} \mathbb{P}(z_{1,2} = a_{1,2})$

$$= \Big(\sum_{a_1 = \pm 1} y^{a_1} \mathbb{P}(z_1 = a_1)\Big)\Big(\sum_{a_2 = \pm 1} y^{a_2} \mathbb{P}(z_2 = a_2)\Big)$$

$$= \left(\tfrac{1}{2} \cdot y + \tfrac{1}{2} \cdot y^{-1}\right)\left(\tfrac{1}{2} \cdot y + \tfrac{1}{2} \cdot y^{-1}\right)$$

So: $G_n(y) = \langle y^{z_1} \rangle^n = \left(\frac{1}{2y} + \frac{y}{2}\right)^n = \frac{1}{(2y)^n}(1 + y^2)^n$

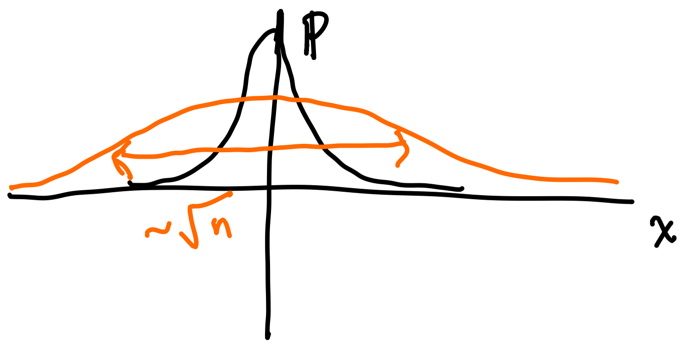
$$= \sum_{k=0}^n \frac{1}{(2y)^n} \binom{n}{k} y^{2k} \rightsquigarrow \quad y^{2k-n} = y^m$$

or $k = \frac{n+m}{2}$

Read off: $\mathbb{P}(x_n = m) = \frac{1}{2^n} \binom{n}{\frac{n+m}{2}} = \frac{1}{2^n} \frac{n!}{\left(\frac{n+m}{2}\right)! \left(\frac{n-m}{2}\right)!}$

Large n? Stirling's approximation: $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

$$\mathbb{P}(x_n = m) \approx \underbrace{\sqrt{\frac{2}{\pi n}}\, e^{-m^2/2n}}_{\approx \text{ Gaussian distribution.}} \qquad \text{if } m \ll n.$$



Restore units: typical value of $x(t) \sim \sqrt{t} \sim \ell\sqrt{t/\tau}$

$$\langle x^2\rangle = 2Dt \quad \text{where } D = \frac{\ell^2}{2\tau} = \text{diffusion const.}$$

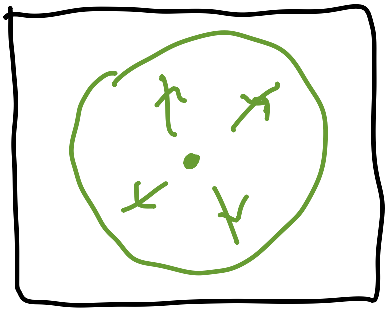
$$\langle x_n^2\rangle = \Big\langle \Big(\sum_{k=1}^{n} z_k\Big)^2 \Big\rangle \underset{\text{linearity of } \langle \cdots \rangle}{=} \sum_{k_{1,2}=1}^{n} \langle z_{k_1} z_{k_2}\rangle$$

$$= \sum_{k_1=1}^{n} \langle z_{k_1}^2\rangle + \sum_{k_1\neq k_2} \langle z_{k_1} z_{k_2}\rangle$$

$$= \sum_{k_1=1}^{n} 1 + \sum_{k_1\neq k_2} \cancelto{0}{\langle z_{k_1}\rangle}\, \cancelto{0}{\langle z_{k_2}\rangle} = n$$

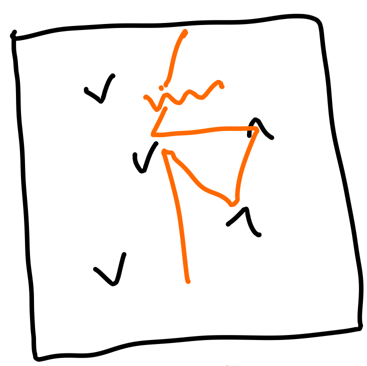
Arrow of time?

macro:



YES

micro:



NO

"forward"

$\mathbb{P}(x_n - x_0 = m)$ vs. "backward" $\mathbb{P}(x_0 - x_n = m)$

$\downarrow$

$$2^{-n}\binom{n}{\frac{n+m}{2}} = 2^{-n}\binom{n}{\frac{n-m}{2}} \quad \checkmark$$

arrow of time comes from <u>knowledge</u> of x_0.