PHYS 7810
Hydrodynamics
Spring 2024

Lecture 10
Incompressible ideal fluids

February 15

Recap: Navier-Stokes for usual liquid/gas:

mass conservation: $\partial_t \rho + \partial_i(\rho v_i) = 0$

mom. —— : $\rho[\partial_t v_i + v_j \partial_j v_i] + \partial_i P - \cancel{[\text{visc.}]} = 0.$

energy —— ~~long~~

Today we will ignore:

1. dissipation
2. energy (heating negligible)
3. ρ fluctuations "negligible": $\rho \approx$ const. $\Rightarrow$ incompressible.

More careful argument: hydro has sound modes

$\omega = \pm v_s k + \ldots$ involve ρ fluctuations.

water: 1500 m/s air: $v_s \sim 330$ m/s

Most "everyday" flows $|v_i| \ll v_s$ ... if

let $\rho = \rho_0 + \frac{1}{v_s^2}\chi$ ← small correction!

keep only leading order as $v_s \to \infty$

$$\cancel{\frac{\partial_t \chi}{v_s^2}} + \rho_0 \boxed{\partial_i v_i} + \cancel{\frac{1}{v_s^2}\partial_i(\chi v_i)} = 0$$

$\partial_i v_i = 0$: incompressible flow.

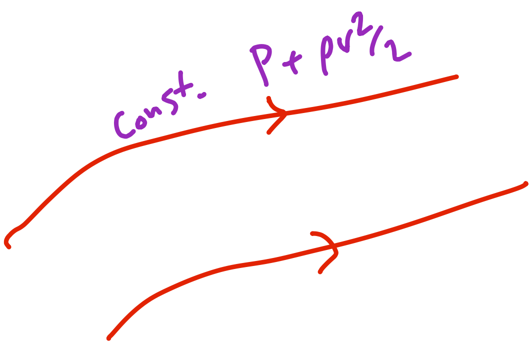
$$\rho_0\left[\partial_t v_i + v_j \partial_j v_i\right] + \cancel{\frac{\partial P}{\partial \rho}}\, \frac{\partial_i \chi}{\cancel{v_s^2}} + \dots = 0.$$

↳ $\cancel{\partial_t v_i} + v_j \partial_j v_i + \frac{\partial_i \chi}{\rho_0} = 0$ (Euler)

How to make further progress? time-independent: $\partial_t = 0$

$$0 = v_j \partial_j v_i + \frac{\partial_i P}{\rho_0} \rightsquigarrow 0 = v_i \frac{\partial_i P}{\rho_0} + \overset{v_i \partial_j v_i}{v_j \partial_j\left(\frac{v^2}{2}\right)}$$

$$0 = \underbrace{v_i \partial_i \left[\frac{P}{\rho_0} + \frac{v^2}{2}\right]}_{\text{Bernoulli's equation}}$$



Bernoulli's equation: $P + \rho_0 \frac{v^2}{2}$ is constant along streamlines = trajectory traced out by $\frac{dx_i(s)}{ds} = v_i(x)$

Useful for calculating forces on objects in flow.

Another simplifying assumption: irrotational flow

$\nabla \times \vec{v} = \vec{0}$ (3d) $[\partial_i v_j - \partial_j v_i = 0] \rightsquigarrow v_i = \partial_i \Phi$

Given incompressibility: $\partial_i v_i = 0 = \underbrace{\nabla^2 \Phi}_{\text{Laplace equation.}}$

Work in $d=2$ spatial dimensions...

Thm: solutions to Laplace correspond to holomorphic functions

[real part of]: $w(\underbrace{x+iy}_{z})$

holomorphic $\to$ no singularity $(1/z, \sqrt{z}\ldots)$ in domain

Claim: $\Phi(x,y) = \mathrm{Re}[w(x+iy)]$ ← real part of

$$(\partial_x^2+\partial_y^2)\Phi = 0 = \underbrace{(\partial_x - i\partial_y)}_{2\,d/dz}\underbrace{(\partial_x+i\partial_y)}_{2\frac{d}{d\bar{z}}}\Phi$$

$2\frac{d}{d\bar{z}}$ = annihilates $w(z)$

Write: $w(z) = \Phi(x,y) + i\Psi(x,y)$

Φ: velocity potential; Ψ: stream function

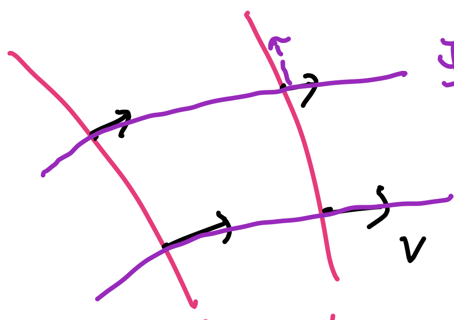
$$v_x = \partial_x\Phi \qquad v_y = \partial_y\Phi$$

$$(\partial_x + i\partial_y)(\Phi + i\Psi) = 0 = [\partial_x\Phi - \partial_y\Psi] + i[\partial_y\Phi + \partial_x\Psi]$$

$$v_x = \partial_y\Psi \qquad v_y = -\partial_x\Psi$$

$$\frac{dw}{dz} = \frac{1}{2}(\partial_x - i\partial_y)(\Phi + i\Psi) = v_x - iv_y$$

For irrotational + incompressible:

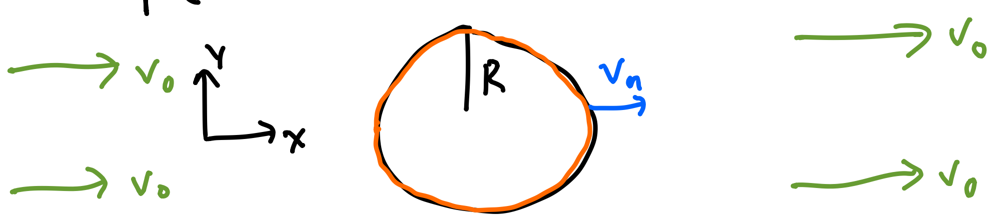


Ψ const. = streamlines

$\downarrow$

Bernoulli: $P + \frac{\rho v^2}{2}$ const. along streamlines.

Example 1: Flow around disk



Boundary conditions?

$v_n = 0$: fluid doesn't flow into disk.

Look for $w(z)$ compatible w/ boundary conds:

As $z \to \infty$:

$w(z) \approx v_0 z$

$\frac{dw}{dz} = v_0 - i0$

$z = R$: $v_n = 0$

streamline: $r = R$

enforces $v_n = 0$:

$\vec{v} \parallel \nabla \Psi$

$w(z) = v_0 \underbrace{re^{i\theta}}_{x+iy} + \frac{c}{\underbrace{re^{i\theta}}}$:

$w(z) = v_0 z + \frac{c}{z}$

singular at $z=0$, fluid flow is domain $|z| \geq R$.

$\Psi = \mathrm{Im}(w) = v_0 r \sin\theta - \frac{c}{r}\sin\theta$

BC at $r=R$: $v_0 R = \frac{c}{R}$

or $c = R^2 v_0$.

$\frac{dw}{dz} = v_x - iv_y \ldots$

What's force acting on disk? Use Bernoulli:

$\underbrace{P + \frac{\rho v^2}{2} = \text{const.}}$ along streamlines $\to$ $r=R$!

Need to evaluate:

$$F_x \hat{x} + F_y \hat{y} = -\int_{r=R} ds\ \hat{n} P = -\int \underbrace{ds}_{R\,d\theta}\ \hat{n}\left(\cancel{\text{const.}} - \frac{\rho v^2}{2}\right)$$

$$= -\int R\,d\theta\,(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})\,\frac{\rho}{2}(v_x^2 + v_y^2)$$

$$F_x - iF_y = \frac{\rho}{2}\int R\,d\theta\,(\cos\theta - i\sin\theta)(v_x + iv_y)\underbrace{(v_x - iv_y)}_{dw/dz}$$

Notice that along streamline: $\vec{v} \parallel$ tangential vector $\hat{\theta}$

$\hat{\theta} = \cos\theta\,\hat{y} - \sin\theta\,\hat{x}$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} \parallel \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} v_x \\ v_y \end{pmatrix} \perp \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$(v_x + iv_y)(\cos\theta - i\sin\theta) = (v_x - iv_y)(\cos\theta + i\sin\theta)\cdot(-1)$$

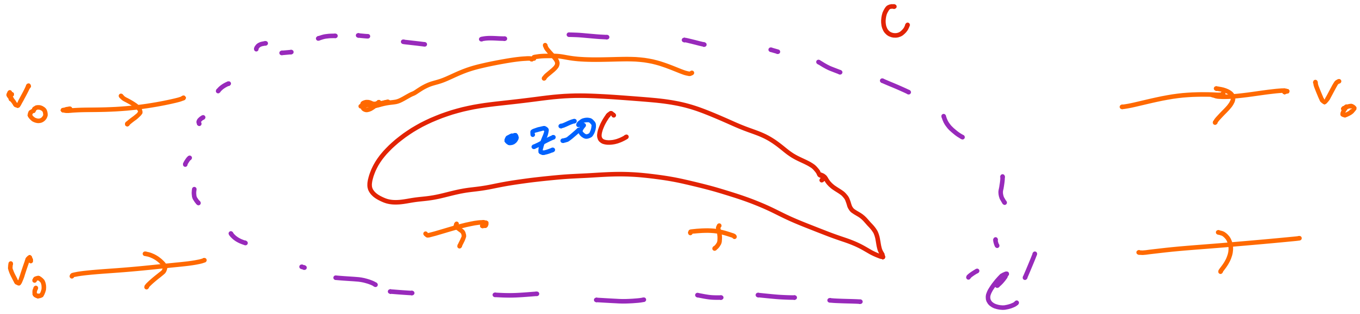
So $$F_x - iF_y = -\int_{r=R} R\,d\theta\,\underbrace{(\cos\theta + i\sin\theta)}_{-i\,dz}\,\underbrace{(v_x - iv_y)^2}_{\left(\frac{dw}{dz}\right)^2}\,\frac{\rho}{2}$$

$$= \frac{i\rho}{2}\int_{r=R} dz\left(\frac{dw}{dz}\right)^2 = \frac{i\rho}{2}\int dz\left(v_0 - \frac{R^2}{z^2}\right)^2 = 0$$

(no simple pole for $|z| < R$).

No forces!

Blasius Theorem: $F_x - iF_y = \frac{i\rho}{2}\int_C dz\left(\frac{dw}{dz}\right)^2$ holds in general



Idea: to get a net force on wing, need $\int_C dz\left(\frac{dw}{dz}\right)^2 \neq 0 \ldots$

$\left(\frac{dw}{dz}\right)^2$ needs simple pole $\left(\frac{1}{z}\right)$ inside C.

At $z \to \infty$: $w(z) = v_0 z - \frac{i\Gamma}{2\pi}\log z + \cdots$

$$\Downarrow$$

$$\frac{dw}{dz} = v_0 - \frac{i\Gamma}{2\pi z} + \mathcal{O}\left(\frac{1}{z^2}\right)$$

If no singularities outside $C \ldots$ $\quad \int_C = \int_{C'}$

Evaluate at large z:

$$F_x - iF_y = \frac{i\rho}{2}\int dz \, \underbrace{\left(v_0 - \frac{i\Gamma}{2\pi z} + \cdots\right)^2}_{v_0^2 - \frac{i\Gamma}{\pi z}v_0 + \cdots} = -i\rho v_0 \Gamma$$

Kutta-Zhukovsky: lift force $F_y = \rho v_0 \Gamma$

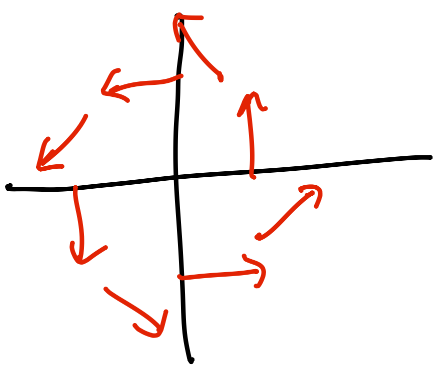
$\downarrow$

when does $\Gamma \neq 0$?

Example 2: $w(z) = -\frac{i\Gamma}{2\pi}\log z$

$\downarrow$

$$v_x = \frac{\Gamma \cdot (-y)}{2\pi(x^2+y^2)} \qquad v_y = \frac{\Gamma x}{x^2+y^2}$$



flow pattern of vortex!