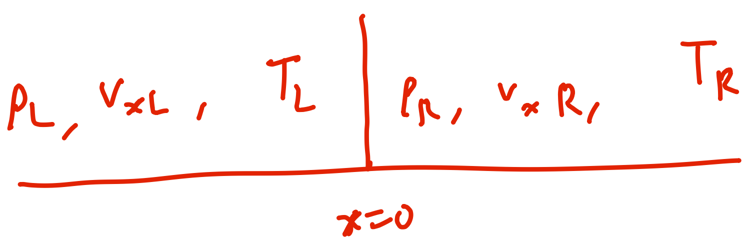
PHYS 7810
Hydrodynamics
Spring 2024

Lecture 11
Shock and rarefaction waves

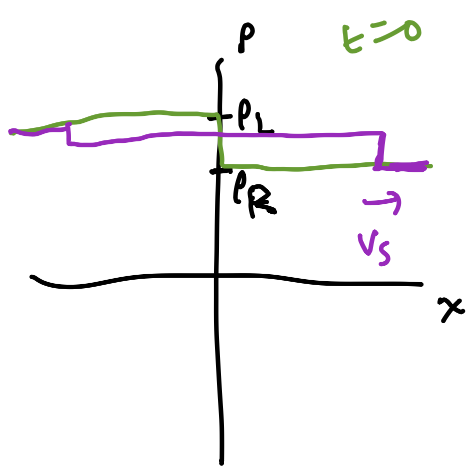
February 20

Today: Riemann problem: at $t=0$

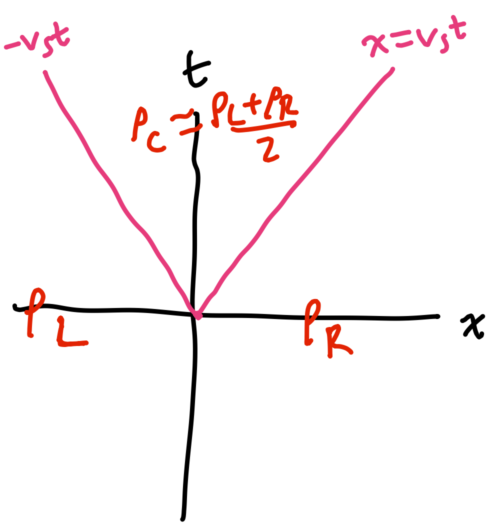


What happens at $t>0$?
Assume fluid is dissipationless (revisit: HW3)

Suppose $T_L = T_R + \delta T$, $\rho_L = \rho_R + \delta\rho$, etc...
use linear response!



in linear response, $\rho(x,t)$ discontinuous



Goal: nonlinear solution of Riemann problem.

Ideas: ① shock wave: discontinuous!

② rarefaction wave: continuous!

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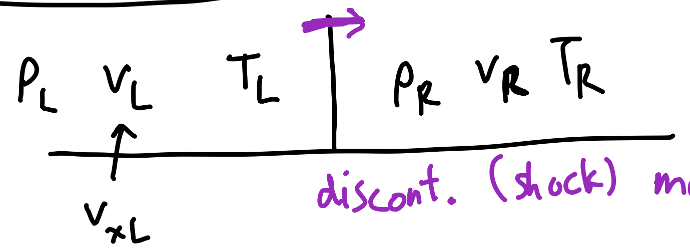
①



discont. (shock) moves at velocity v_{sh}

<u>Intuition:</u> ρ_R, v_R, T_R (mostly) fixed by conservation laws!

If Galilean-inv. fluid ... boost to frame where shock is static!

mass: $\cancel{\partial_t \rho} + \partial_x J_x = 0 \longrightarrow J_x = \text{const.}$

static

$v_R = \tilde{v}_R + v_{sh}$

in this special frame: $J_{xL} = \rho_L \tilde{v}_L = J_{xR} = \rho_R \tilde{v}_R$

momentum: $T_{xx,L} = T_{xx,R} \longrightarrow P_L + \rho_L \tilde{v}_L^2 = P_R + \rho_R \tilde{v}_R^2$

energy: $\tilde{v}_L\left[\varepsilon_L + P_L + \frac{1}{2}\rho_L \tilde{v}_L^2\right] = \tilde{v}_R\left[\varepsilon_R + P_R + \frac{1}{2}\rho_R \tilde{v}_R^2\right]$

Solve these nonlinear equations? Neglect energy conservation...

$\rho_L \tilde{v}_L = \rho_R \tilde{v}_R$ and $P(\rho_L) + \rho_L \tilde{v}_L^2 = P(\rho_R) + \rho_R \tilde{v}_R^2$

$\hookrightarrow \tilde{v}_R = \frac{\rho_L}{\rho_R}\tilde{v}_L$

$\hookrightarrow P(\rho_L) + \rho_L \tilde{v}_L^2 = P(\rho_R) + \frac{\rho_L^2}{\rho_R}\tilde{v}_L^2$

Solve for ρ_R.

Unboost frame:

$v_R - v_{sh} = \frac{\rho_L}{\rho_R}(v_L - v_{sh})$ and $P(\rho_L) + \rho_L (v_L - v_{sh})^2$

$= P(\rho_R) + \frac{\rho_L^2}{\rho_R}(v_R - v_{sh})^2$

Given ρ_L, v_L ... if ρ_R takes specific value ↑ for given v_R ... might simultaneously solve both equations.

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② Look for a smooth solution:

$$\rho(x,t) = \rho\left(\frac{x}{t}\right) \qquad \text{and} \qquad v(x,t) = v\left(\frac{x}{t}\right)$$

call this ξ

Now: $\partial_t \rho + \partial_x J_x = \partial_t \rho + \partial_x(\rho v) = 0$

$$-\frac{x}{t^2}\frac{d\rho}{d\xi} + \frac{v}{t}\frac{d\rho}{d\xi} + \frac{\rho}{t}\frac{dv}{d\xi} = 0. \rightarrow -\xi\rho' + (\rho v)' = 0.$$

Similarly: $-\xi(\rho v)' + (P(\rho) + \rho v^2)' = 0.$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\xi + v & \rho \\ P'(\rho) + v(v-\xi) & -\xi\rho + 2\rho v \end{pmatrix}\begin{pmatrix} \rho' \\ v' \end{pmatrix}$$

The equations have a solution (I care about) if $\det(2\times2) = 0$.

$$\det(2\times 2) = \rho(v-\xi)(2v-\xi) - \rho\left[P'(\rho) + v(v-\xi)\right] = 0$$

$$\downarrow$$

$$P'(\rho) = (v-\xi)^2 \longrightarrow \text{solved for } \rho(v,\xi)$$

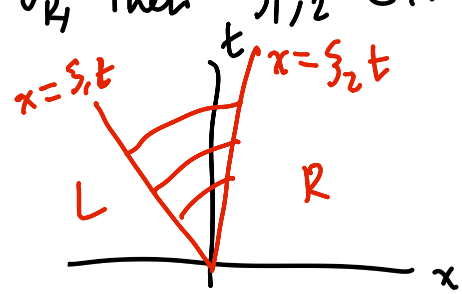
$(v-\xi)\rho(v,\xi)' + \rho(v,\xi)v' = 0. \longrightarrow$ solve for $v(\xi)$, then $\rho(\xi)$.

Idea: suppose start w/ ρ_L, v_L $(\xi \to -\infty)$.

If rarefaction wave exists ... to ρ_R, v_R, then $\xi_{1,2}$ exist;

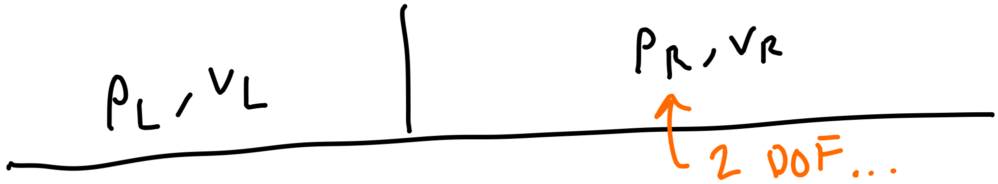
$$\rho(\xi_1) = \rho_L \qquad \rho(\xi_2) = \rho_R$$

$$v(\xi_1) = v_L \qquad v(\xi_2) = v_R$$

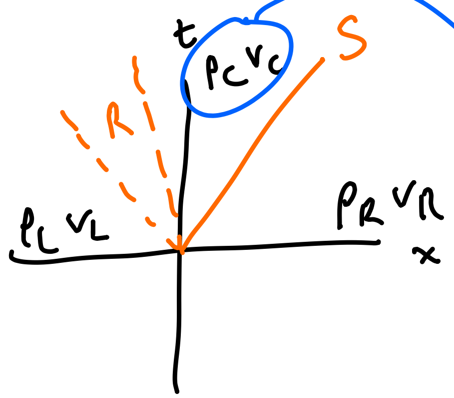


Fix ξ_1?

Recap: solve Riemann problem... at $t=0$.



2 DOF...
1 for shock 1 rarefaction.



Existence of "C" region adds enough "DoF" to solve Riemann problem.

Warning: Riemann problem (ideal hydro) has many solutions.

Which is physical? Test physicality by looking @ 2nd law...

if shock solution... entropy production must be ≥ 0.

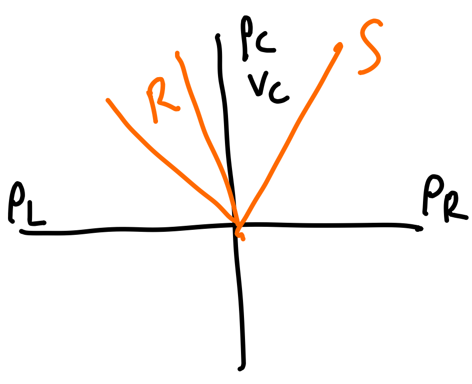
$\hookrightarrow$ (HW3)

(Entropy production negligible in rarefaction.)

Example: gas dynamics: $P(\rho) = A\rho^\gamma \rightarrow A, \gamma = \text{const}.$

Solve Riemann w/ $v_L = v_R = 0$, but $\rho_L > \rho_R$

Claim:



$\rho_L > \rho_C > \rho_R$ and $v_C > 0$.

First, deduce rarefaction wave:

$$(v-\xi)^2 = P'(\rho) = A\gamma \cdot \rho^{\gamma-1} \quad \text{or} \quad \rho = \underbrace{(A\gamma)}_{A\gamma=1\ !!}{}^{-\frac{1}{\gamma-1}} (v-\xi)^{\frac{2}{\gamma-1}}$$

$$(v-\xi)\rho' + \rho v' = 0$$

$$\hookrightarrow 0 = (v-\xi)(v'-1)\frac{2}{\gamma-1}(v-\xi)^{-1+\frac{2}{\gamma-1}} + (v-\xi)^{\frac{2}{\gamma-1}} v'$$

↳ $0 = \frac{\gamma+1}{\gamma-1} v' - \frac{2}{\gamma-1}$ or $v(\xi) = v(\xi_1) + (\xi - \xi_1)\frac{2}{\gamma+1}$

If rarefaction starts at ρ_L & $v_L = 0$:

$$\rho_L = (0 - \xi_1)^{\frac{2}{\gamma-1}} \quad \text{or} \quad \boxed{\xi_1 = -\rho_L^{\frac{\gamma-1}{2}}.}$$ fix ξ_1.

ξ_2 undetermined ... fix ρ_c, v_c:

$$\boxed{v_c = \frac{2}{\gamma+1}(\xi_2 - \xi_1) \qquad \rho_c = \rho_L \left| \frac{2}{\gamma+1} + \frac{\xi_2}{\xi_1}\frac{\gamma-1}{\gamma+1} \right|^{\frac{2}{\gamma-1}}}$$

Fix ξ_2 by finding ρ_c, v_c that lead to shock to $\rho_R, v_R = 0$.

$$\rho_R(-v_{sh}) = \frac{\rho_c}{\rho_R}(v_c - v_{sh}) \quad \text{and} \quad P(\rho_c) + \rho_c(v_c - v_{sh})^2 = P(\rho_R) + \rho_R v_{sh}^2$$

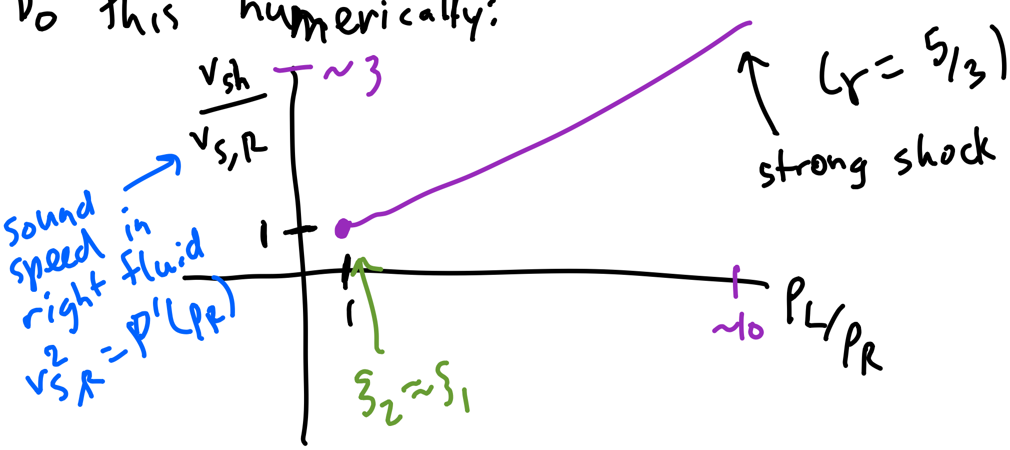
↳ $v_{sh} = \frac{\rho_c v_c}{\rho_c - \rho_R}$

A

$$\boxed{\frac{1}{\gamma}(\rho_R^\gamma - \rho_c^\gamma) = \frac{\rho_R \rho_c}{\rho_R - \rho_c} v_c^2}$$

simultaneously solve nonlinear eq. for ρ_c & v_c.

Do this numerically:



HW3: check that entropy produced across shock front.

Our sol'n above (w/ shock) has $\rho(x/t) = \rho(\xi)$, consequence of ideal hydro: $\partial_t \rho + \partial_x(\rho v)$ $(-D\partial_x^2 \rho)$ → diss. breaks rescaling sym.