PHYS 7810
Hydrodynamics
Spring 2024

Lecture 12
Vortex dynamics

February 22

Today: dynamics of vorticity in Galilean-invariant fluids:

$$\omega_i = \varepsilon_{ijk}\partial_j v_k \qquad [\vec{\omega} = \nabla\times\vec{v}].$$

Assume flow incompressible: $\nabla\cdot\vec{v} = 0 = \partial_i v_i$

Navier-Stokes equation: const. (assume) kinematic viscosity (air: $\nu \sim 10^{-5}\,\frac{m^2}{s}$)

$$\partial_t v_i + v_j\partial_j v_i + \frac{1}{\rho}\partial_i P = \frac{\eta}{\rho}\partial_j\partial_j v_i = \nu\,\partial_j\partial_j v_i$$

$$\partial_t \omega_i + \varepsilon_{ijk}\partial_j\left[v_l\partial_l v_k + \cancel{\frac{1}{\rho}\partial_k P}\right] = \nu\,\partial_l\partial_l\omega_i$$

$$\partial_t\omega_i + v_l\partial_l\omega_i + \underbrace{(\partial_l v_k)}_{\partial_k v_l + \varepsilon_{lkm}\omega_m}\varepsilon_{ijk}\partial_j v_l = \nu\,\partial_l\partial_l\omega_i$$

$$\hookrightarrow \cancel{\varepsilon_{ijk}(\partial_k v_l)(\partial_j v_l)} + \underbrace{\varepsilon_{ijk}\varepsilon_{lkm}\omega_m\partial_j v_l}_{-\omega_m\partial_m v_i + \cancel{\omega_i\partial_m v_m}}$$

Conclude:

$$\partial_t\omega_i + v_l\partial_l\omega_i = \omega_m\partial_m v_i + \nu\,\partial_j\partial_j\omega_i$$

Limit of 2d flow: $v_z = 0$ and $\partial_z = 0$.

$$\omega_z = \partial_x v_y - \partial_y v_x.$$

$$\underbrace{\partial_t \omega_z + \vec{v}\cdot\nabla\omega_z}_{\text{each vortex is convected}} = \underbrace{\nu\nabla^2\omega_z}_{\text{vorticity diffuses}}$$

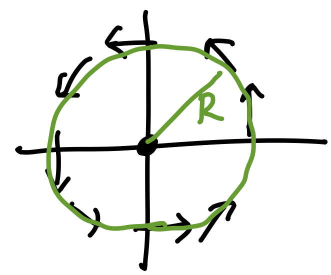
[but $\vec{v}$ depends on ω_z].

Example 1: isolated "point vortex":

from lec 10: $v_x = \frac{\Gamma}{2\pi}\frac{-y}{x^2+y^2}$ $\qquad v_y = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$

$\hookrightarrow$ $v_\theta = \frac{\Gamma}{2\pi r}$ $\qquad v_r = 0.$ $\qquad$ (polar coord)

$$\omega_z = \partial_x v_y - \partial_y v_x = \Gamma\,\delta(x)\,\delta(y)$$



$\hookrightarrow$ $\oint \vec{v}\cdot d\vec{s} = \frac{\Gamma}{2\pi R}\cdot 2\pi R = \Gamma$

$= \int \omega_z\, dx\, dy$ by Stokes' Thm.

Incompressible: $\nabla\cdot\vec{v} = 0$

$$\partial_t\omega_z + \vec{v}\cdot\nabla\omega_z = 0 \qquad \frac{\Gamma}{2\pi r}\frac{\partial\omega_z}{\partial\theta} = 0.$$

static point vortex is a solution.

Add viscosity? $\partial_t\omega_z + \cancel{\vec{v}\cdot\nabla\omega_z} = \nu\nabla^2\omega_z$

rotational symmetry

$$\omega_z(r,\theta,t) = \frac{\Gamma}{4\pi\nu t}e^{-r^2/4\nu t} \qquad \text{(cf lec. 2)}$$

$\hookrightarrow$ Expect: $\vec{v} = v_\theta(r,t)\,\hat{\theta}$

$$\nabla\times\vec{v} = \omega_z = \frac{1}{r}\partial_r(r v_\theta)$$

Integrate to get: $\displaystyle\int_r^\infty dr'\, r' \omega_z(r') = r' v_\theta(r')\Big|_r^\infty$

$$\frac{\Gamma}{2\pi} e^{-r^2/4\nu t} = \frac{\Gamma}{2\pi} - r v_\theta(r)$$

So $\displaystyle v_\theta(r) = \frac{\Gamma}{2\pi r}\left[1 - e^{-r^2/4\nu t}\right].$

Example 2: 2 point vortices, 2d...

Neglect dissipation: $\partial_t \omega_z + \vec v \cdot \nabla \omega_z = 0.$

Suppose $\omega_z = \Gamma_1 \delta(\vec r - \vec r_1(t)) + \Gamma_2 \delta(\vec r - \vec r_2(t))$.

$$-\Gamma_1 \dot{\vec r}_1 \cdot \nabla \delta(\vec r - \vec r_1) - \Gamma_2 \dot{\vec r}_2 \cdot \nabla \delta(\vec r - \vec r_2) + \Gamma_1 \vec v(\vec r_1) \cdot \nabla \delta(\vec r - \vec r_1) + \Gamma_2 \vec v(\vec r_2) \cdot \nabla \delta(\vec r - \vec r_2) = 0$$

↳ $\dot{\vec r}_1 = \underbrace{\vec v(\vec r_1)}$ and $\dot{\vec r}_2 = \vec v(\vec r_2)$

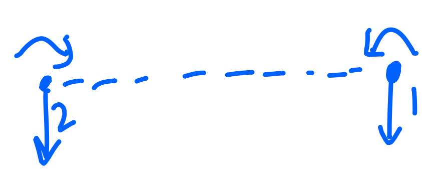
contributions to $\vec v$ from other vortices.

$$\frac{d\vec r_1}{dt} = \vec v_2(\vec r_1) = \frac{\Gamma_2}{2\pi} \frac{\hat z \times (\vec r_1 - \vec r_2)}{|\vec r_1 - \vec r_2|^2} \quad \text{and} \quad \frac{d\vec r_2}{dt} = \frac{\Gamma_1}{2\pi} \frac{\hat z \times (\vec r_2 - \vec r_1)}{|\vec r_1 - \vec r_2|^2}.$$

Suppose $\Gamma_1 = \Gamma_2 > 0$:

point vortices will move in a circle.

Suppose $\Gamma_1 = -\Gamma_2$



"vortex dipole" moves at const. velocity.

Generalize to n point vortices: vortex $\alpha = 1, \ldots, n$

$$\frac{d\vec{r}_\alpha}{dt} = \sum_{\beta (\neq \alpha)} \frac{\Gamma_\beta}{2\pi} \frac{\hat{z} \times (\vec{r}_\alpha - \vec{r}_\beta)}{|\vec{r}_\alpha - \vec{r}_\beta|^2} .$$

Nice math problem! $\longrightarrow$ Hamiltonian mechanics!

$$H = -\sum_{\alpha \neq \beta} \frac{\Gamma_\alpha \Gamma_\beta}{2\pi} \log |\vec{r}_\alpha - \vec{r}_\beta| \quad \text{with} \quad \{x_\alpha, y_\beta\} = \Gamma_\alpha \delta_{\alpha\beta} .$$

canonical conj!

H is conserved... as is $\sum \Gamma_\alpha x_\alpha$, $\sum \Gamma_\alpha y_\alpha$, $\sum \Gamma_\alpha (x_\alpha^2 + y_\alpha^2)$

Physics:
- superfluid vortex dynamics
  - ↳ quantized $\Gamma = \pm \hbar / m$
- 2d charged particles in strong B-field (Lowest Landau level)

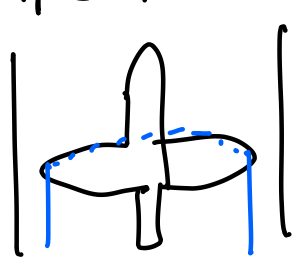
Example 3: point vortex near wall

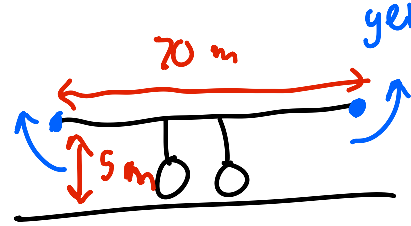
wall: $v_y(y=0) = 0.$

$$v_y = \underbrace{\frac{-\Gamma x}{x^2 + (y-a)^2}} - \underbrace{\frac{-\Gamma x}{x^2 + (y+a)^2}}$$

indeed: $v_y(y=0) = 0$.

Example 4: "wingtip vortex" of planes... needed to generate lift!





lead to dangerous flow for other planes.

$\to$ viscous dissipation? (air: $\nu \sim 10^{-5}\,m^2/s$)

diffuse $\sim 5m \sim \sqrt{\nu t}$ $\rightsquigarrow$ $t \sim 2.5 \times 10^6\,s$

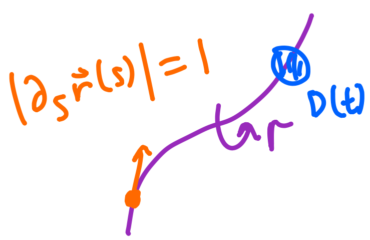
$\to$ vortex motion:

estimate $\Gamma \sim v_{plane} \cdot \underset{\nearrow 0.5m}{W}$: $V_{vortex} \sim \frac{\Gamma}{h} \sim 5\ m/s$ (too big)

$(t \sim 1\,s)$

Reality: ~ 1 min (per FAA).

Now let's turn to 3d vortex line motion:



parameterize by $\vec{r}(s)$

Analogous to Biot-Savart:

$$\vec{v}(\vec{x}) = \frac{\Gamma}{4\pi}\int ds\, \frac{\partial_s \vec{r} \times (\vec{x} - \vec{r}(s))}{|\vec{x} - \vec{r}(s)|^3}$$

Claim: $\frac{\partial}{\partial t}\vec{r} = \vec{v}(\vec{r}, t)$.

To justify: $\frac{d}{dt}\oint_{\partial D(t)} \vec{v}(\vec{r},t)\cdot d\vec{r} \overset{\text{Stokes}}{=} \frac{d}{dt}\int_D \vec{\omega}\cdot d\vec{A}$ $\sim$ "$\frac{d}{dt}\Gamma$"

$$= \oint_{\partial D} [\partial_t \vec{v} + \vec{v}\cdot\nabla\vec{v}]\cdot d\vec{r} + \oint_{\partial D} \vec{v} \cdot \overbrace{\frac{d\vec{r}}{dt}}^{d\vec{v}}$$

$$= \oint_{\partial D} \left(-\frac{1}{\rho} \nabla P + \cancel{\nu \nabla^2 \vec{v}} \right) \cdot d\vec{r} + \oint_{\partial D} d\left(\frac{v^2}{2}\right)$$

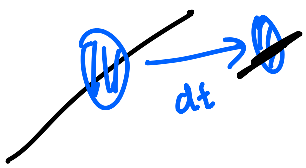
ignore diss

$$= 0 \quad + 0 \qquad \left(\oint d\vec{r} \cdot \nabla f = 0 \right)$$

Kelvin's Circulation Thm:

$$\frac{d}{dt} \int_{\partial D(t)} \vec{v} \cdot d\vec{s} = 0 :$$

vortex line lasts forever...
convects along w/ flow



circulation pierces this disk, so
vortex line obeys

$$\frac{\partial \vec{r}}{\partial t} = \vec{v}(\vec{r}, t)$$