PHYS 7810
Hydrodynamics
Spring 2024

Lecture 13
Examples of incompressible viscous flow

February 24

Recall: Navier-Stokes equations for incompressible fluid $(\nabla \cdot \vec{v} = 0)$

upon assuming $\rho, T \approx$ const.

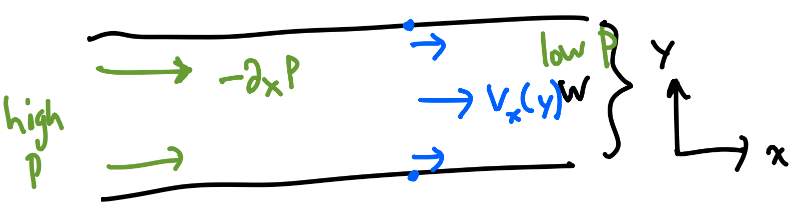
$$\partial_t v_i + v_j \partial_j v_i + \partial_i \frac{P}{\rho} = \frac{\eta}{\rho} \partial_j \partial_j v_i = \nu \partial_j \partial_j v_i$$

Today: examples of exact solutions.

Example 1: Pipe flow (Poiseuille flow)

Assume no-slip boundary cond.

$\vec{v} = \vec{0}$ at boundaries.



Look for static $\partial_t = 0$: and $\partial_x v_i = 0$

Ansatz: $v_x(y)$

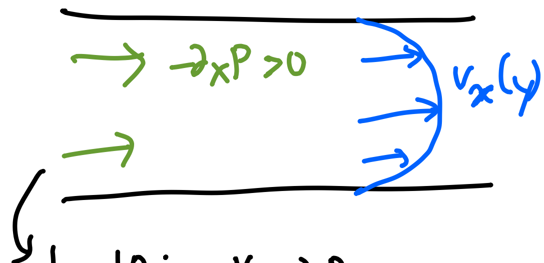
$$\cancel{v_x \partial_x v_x} + \underbrace{\frac{\partial_x P}{\rho}}_{\text{const.}} = \nu \partial_j \partial_j v_x = \nu \partial_y^2 v_x$$

integration const.

$$v_x(y) = \frac{\partial_x P}{2\eta} y^2 + c_1 y + c_2$$

Set $v_x(y=0) = 0 = v_x(y=w)$:

$$c_2 = 0 \qquad\qquad c_1 = -\frac{\partial_x P}{2\eta} w$$



Lec 10: $v_s \to \infty$

$P \approx P(\rho) + v_s^2 \delta_\rho + \dots$

$\partial_x P \sim \frac{1}{v_s^2} \partial_x \delta_\rho$

Thus: $$v_x(y) = -\frac{\partial_x P}{2\eta} w(x-w)$$

What's mass flow?

$$Q = \int_0^w dy \cdot \rho v_x(y) = \frac{|\partial_x P|}{2\eta} \int_0^w dy \, (wy - y^2) = \frac{|\partial_x P|}{2\eta}\left(\frac{w^3}{2} - \frac{w^3}{3}\right) = \frac{|\partial_x P|}{12\eta} w^3$$

$\hookrightarrow w^4$ (3d)

e.g. circulatory system:

P_i

P_f

How does fluid flow thru network?

Analogous to resistor network:

"voltage": pressure P scalar, well-defined at junction ($\sum \Delta P = 0$)

"current": $Q_1 = Q_2 + Q_3$, or $Q_{in} = Q_{out}$ at junction

$$I = \frac{\Delta V}{R} \rightsquigarrow Q = \frac{\Delta P}{R}, \text{ so } \frac{1}{R} = \frac{1}{L} \times \frac{w^3}{12\eta}$$

L: length

since $\Delta P = L \cdot |\partial_x P|$

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w/o so many symmetries, exact nonlinear sol'n won't exist...

static sol'ns: $$\underbrace{v_j \partial_j v_i} + \frac{\partial_i P}{\rho} = \underbrace{\nu \partial_j \partial_j v_i}$$

Useful to non-dimensionalize:

$$\tilde{v}_i = \frac{v_i}{\bar{v}} \;\leftarrow \text{"typical velocity"}, \qquad l^{-1} \frac{\partial}{\partial \tilde{x}_i} = \frac{\partial}{\partial x_i}$$

l ← typical length

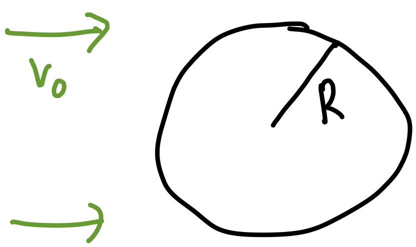
Compare typical size of 2 velocity-dep. terms:

$$\frac{\text{"}\, v_j \partial_j v_i \,\text{"}}{\nu \partial_j \partial_j v_i} \sim \frac{\bar{v}^2/\ell}{\nu \bar{v}/\ell^2} = \frac{\bar{v}\ell}{\nu} = R \quad \text{(Reynolds number) (Re)}$$

↳ ~ up to factor of ~2, ambiguous

- $R \ll 1$: viscous terms dominate (creep flow) ⇝ bacteria...
- $R \gg 1$: inviscid flow (convective dominate) ⇝ airplanes (lec 14–15)

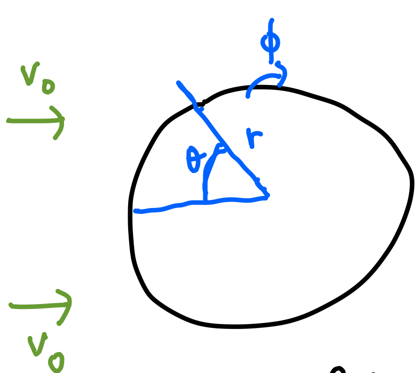
Example 2: drag force on a sphere.



(cf lec 10: in irrotational flow ($R \to \infty$?) $\vec{F} = \vec{0}$.)

Calculate drag forces in opposite limit $R \ll 1$:

$$\cancel{v_j \partial_j v_i} + \frac{\partial_i P}{\rho} = \nu \partial_j \partial_j v_i \xrightarrow{\text{curl}} \nu \nabla^2 (\nabla \times \vec{v}) = \vec{0} = \nu \nabla^2 \vec{\omega}$$



Flow independent of ϕ

Ansatz: $\vec{v} = \nabla \times \vec{A} = \nabla \times (\underbrace{\psi(r,\theta)}_{\text{"stream function"}}\, \hat{\phi})$

EOM: $\nu \nabla^2 (\nabla^2 \vec{A}) = 0$ or $\nabla^2 \nabla^2 (\psi \hat{\phi}) = 0.$

$$\downarrow$$

$$\left(\nabla^2 - \frac{1}{r^2 \sin^2\theta}\right)^2 \psi = 0.$$

Boundary conditions: ① $\vec{v} \to \vec{v}_0$ as $r \to \infty$; $\vec{v}_0 = -v_0 \hat{z} = -v_0(\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta})$

② no-slip at $r = R$ $(\vec{v} = \vec{0})$

① $\vec{v}\cdot\hat{\theta} = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$ as $r\to\infty$: $\frac{1}{r}\frac{\partial}{\partial r}(r\psi) \to v_0 \sin\theta$

Deduce as $r\to\infty$: $\psi \to \frac{1}{2} v_0 r \sin\theta$

By spherical symmetry... expand ψ into spherical harmonics

$$\downarrow$$

Ansatz: $\psi(r,\theta) = \sin\theta \cdot f(r)$

$$\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)^2 (\sin\theta \cdot f(r)) = 0$$

$$\downarrow$$

$$\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{2}{r^2}\right)\underbrace{\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{2}{r^2}\right)f}_{g} = 0$$

$\frac{1}{r^2}\frac{d}{dr}(r^2 g') = \frac{2}{r^2}g \to$ Ansatz: $g(r) = r^\alpha$?

$$\downarrow$$

$$g(r) = c_1 r + \frac{c_2}{r^2}$$

$$= \left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{2}{r^2}\right)f$$

from "inhomogeneous solution"

So: $f(r) = \tilde{c}_1 r + \frac{\tilde{c}_2}{r^2} + \underbrace{\tilde{c}_3 + \cancel{\tilde{c}_4 r^3}}$.

$\hookrightarrow \vec{v}\sim r^2$ at large r: forbidden

$$\vec{v} = \nabla\times(\psi\hat{\phi}) = \frac{2f}{r}\cos\theta\,\hat{r} - \frac{1}{r}\frac{d}{dr}(rf)\sin\theta\,\hat{\theta}$$

If $\vec{v}\to -v_0\hat{z}$ at large r: $\tilde{c}_1 = -\frac{1}{2}v_0$

No-slip at $r=R$:

$$f(R) = 0 = -\frac{v_0 R}{2} + \frac{\tilde{c}_2}{R^2} + \tilde{c}_3 \qquad [v_r = 0]$$

$$0 = (rf)'\big|_{r=R} = -v_0 R - \frac{\tilde{c}_2}{R^2} + \tilde{c}_3 \qquad [v_\theta = 0]$$

Solve: $\tilde{c}_3 = \frac{3}{4} v_0 R, \quad \tilde{c}_2 = -\frac{v_0 R^3}{4}.$

What are drag forces acting on sphere?

$$\frac{\text{force}}{\text{area}} = \frac{\frac{d}{dt} \int_{\text{patch}} \text{momentum}}{\text{area}} = \frac{\int_{\text{box}} d^3x \, (-\partial_j \tau_{ji})}{\text{area}}$$

$\hookrightarrow - n_j \tau_{ji}$

Since $\tau_{ji} = P \delta_{ji} - \eta (\partial_i v_j + \partial_j v_i)$ (if incompressible)

$$F_i = - \int_{r=R} r^2 \sin\theta \, d\theta \, d\phi \cdot \hat{r}_j \tau_{ji}$$

(by rot. inv. F_z is only non-vanishing component)

$\downarrow$

$$F_z = 2\pi R^2 \int \sin\theta \, d\theta \left[\underbrace{\hat{r} \cdot \hat{z}}_{\cos\theta} P - \eta (\partial_r v_z + \partial_z v_r) \right]$$

$v_z = v_r \cos\theta - \sin\theta \, v_\theta$ and $\partial_z = \cos\theta \, \partial_r - \frac{\sin\theta}{r} \partial_\theta$

$$\partial_r v_z + \partial_z v_r = \cos^2\theta \cdot \partial_r \left(\frac{4f}{r} \right) + \sin^2\theta \left[\cancel{\frac{2f}{r^2}} + \partial_r \left(\frac{1}{r} \partial_r (fr) \right) \right]$$

Evaluate at $r=R$: $f(R) = 0$

Since $f(r) = -\frac{v_0 r}{2} + \frac{3}{4} v_0 R - \frac{v_0 R^3}{4r^2}$: show $\partial_r \left(\frac{f}{r} \right) \Big|_{r=R} = 0.$

$\downarrow$

$$\partial_r v_z + \partial_z v_r = \sin^2\theta \cdot \left(-\frac{3}{2} \frac{v_0}{R} \right)$$

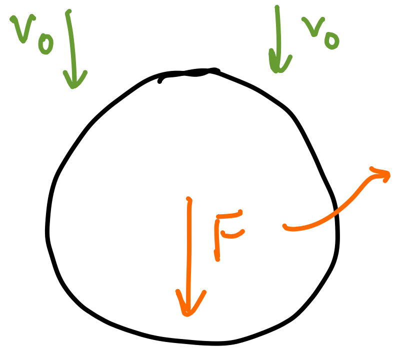
To calculate pressure: $\underbrace{\eta \nabla^2 \vec{v} = \nabla P}_{\text{divergence: } \nabla^2 P = 0}$

Using rotational sym: $P(r,\theta) = \frac{B \cos\theta}{r^2}$ (+ const.)

Evaluate: $B = \frac{3}{2}\eta v_0 R$

So: $$F_z = -2\pi R^2 \int_0^\pi d\theta \sin\theta \left[\cos^2\theta \, \frac{3}{2}\eta \frac{v_0}{R} + \frac{3}{2}\sin^2\theta \, \eta \frac{v_0}{R}\right]$$

$$= -3\pi R v_0 \eta \int_0^\pi d\theta \sin\theta = \underline{-6\pi\eta R v_0}$$



$F \parallel v_0$: drag force

Reynolds #

Flow pattern above assumed $\mathcal{R} \ll 1$; is this consistent?

Use method of dominant balance:

$$\underbrace{\rho \vec{v}\cdot\nabla\vec{v}} + \underbrace{\nabla P} = \underbrace{\eta \nabla^2 \vec{v}}$$ creep flow: dominant?

as $r \to \infty$:

$\vec{v}_0 \cdot \nabla \vec{v}$

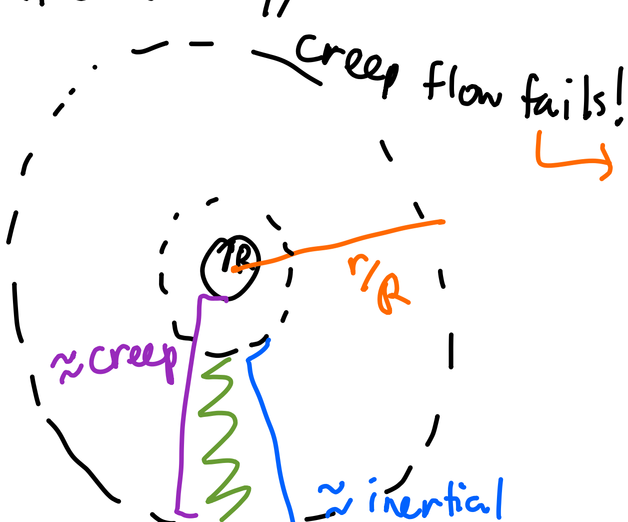
$\sim v_0 \frac{\tilde{C}_3}{r^2}$

as $r \to \infty$: $\nabla P \sim \frac{\eta v_0 R}{r^3}$

falls off more slowly! $r^{-2} \rightarrow$ inconsistent!

Claim: calculation of $\vec{F}$ is accurate... $\vec{v} - \vec{v}_0$ at large r:

Mathematically, solve N-S approx matched asymptotic expansion:



$\hookrightarrow$ Estimate by:

$$\rho \frac{v_0^2 R}{r^2} \gtrsim \frac{\eta v_0 R}{r^3}$$

$$\text{or} \quad r \sim \frac{\eta}{\rho v_0} \sim \frac{R}{\mathcal{R}}$$

both solutions must be good approximations...