PHYS 7810
Hydrodynamics
Spring 2024

Lecture 14
Boundary layers

February 29

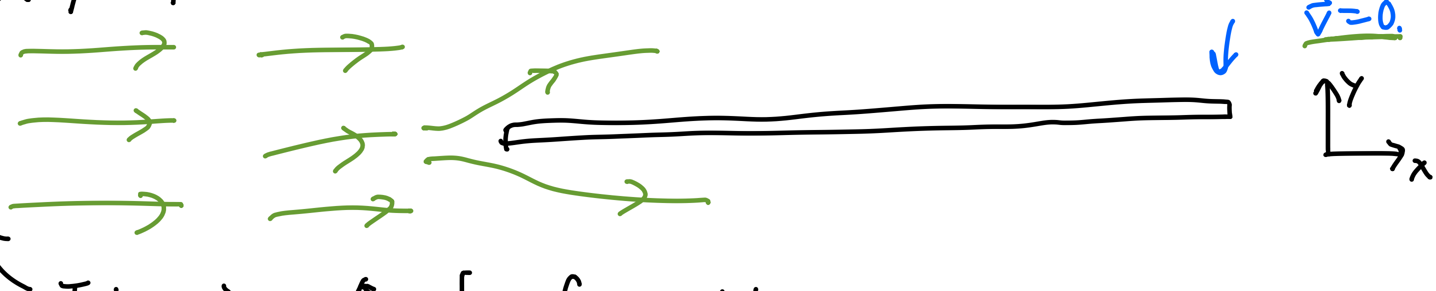
Navier–Stokes equations: incompressible: $\nabla \cdot \vec{v} = 0$

$$\cancel{\partial_t} \vec{v} + \vec{v} \cdot \nabla \vec{v} + \frac{\nabla P}{\rho} = \nu \nabla^2 \vec{v}$$

static (assumption)

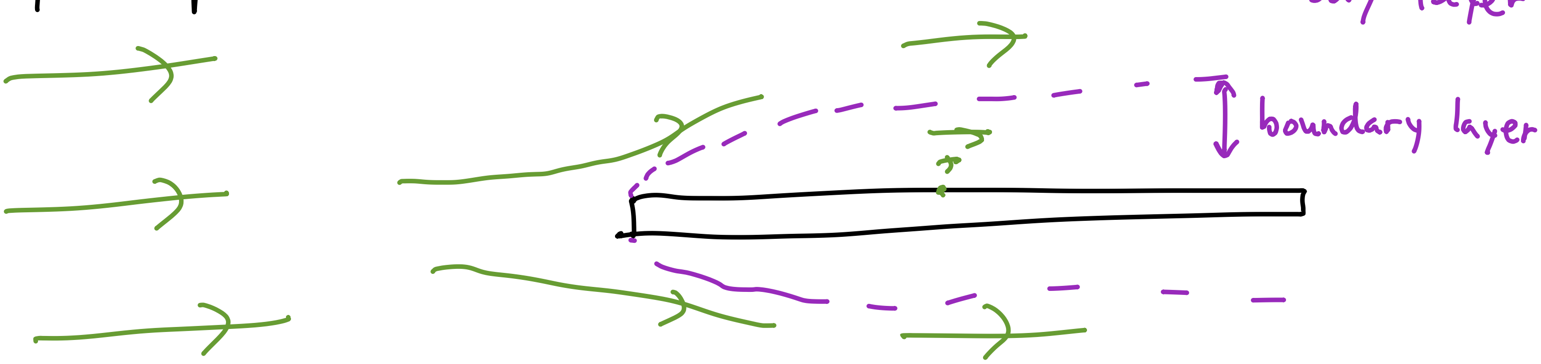
Today: flow around thin plate...

assume no-slip BC: $\vec{v} = \vec{0}$.

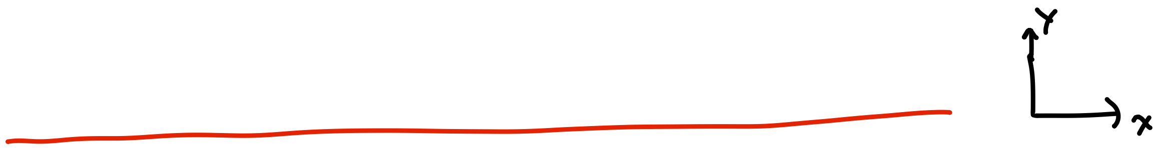


Take $\vec{v} = v_0 \hat{x}$ far from plate

Physical picture: flow is $\approx$ const./homogeneous... outside of bdy layer



Warm-up: infinite plate $\longrightarrow$ $v_x(y\to\infty, t) = f(t)$



$y=0$

Assume: translation invariance in x $(\partial_x = 0)$

incompressible: $\cancel{\partial_x v_x} + \partial_y v_y = 0$

Thus: $v_y = g(x,t) = 0$ by looking at $y=0$.

x-Navier-Stokes:

$$\partial_t v_x + \cancel{v_x \partial_x v_x} + \cancel{v_y \partial_y v_x} = \nu \partial_y^2 v_x + \partial_t f(t)$$

$\frac{\partial_x P}{\rho}$ drives faraway flow.

Take t-Fourier transform: $v_x(t) \to \hat{v}_x(\omega)$

$$-i\omega \hat{v}_x = \nu \partial_y^2 \hat{v}_x - i\omega \hat{f}(\omega)$$

↳ Solved by $\hat{v}_x(y,\omega) = \hat{f}(\omega) + A e^{+\sqrt{-i\omega/\nu}\, y} + B e^{-\sqrt{\frac{-i\omega}{\nu}}\, y}$

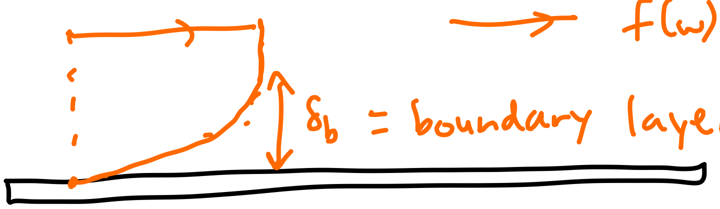
$$\sqrt{-i} = \frac{1-i}{\sqrt{2}} = e^{-i\pi/4}$$

$A=0$ (regularity at $y=\infty$)

$B = -\hat{f}(\omega)$ due to no-slip.

Thus: $\hat{v}_x(y,\omega) = \hat{f}(\omega)\left(1 - e^{\frac{i-1}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}\, y}\right)$ $\leftarrow e^{-y/\delta_b}$ (NB: $\omega > 0$ here)

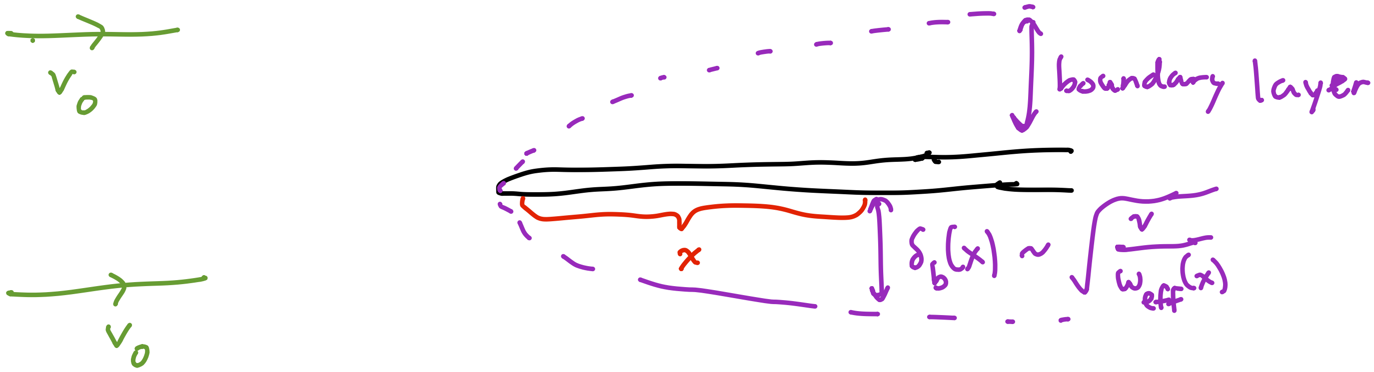
$\longrightarrow \hat{f}(\omega)$



δ_b = boundary layer thickness $= \sqrt{\frac{\nu}{\omega}}$

Outside $y \gg \delta_b$, $\vec{v} \approx v_0 \hat{x}$ (irrotational sol'n, neglecting no-slip)

In our original problem, no explicit t-dep.



By dimensional analysis: $w_{eff}(x) = v_0 / x$

Goal: "perturbative" solution to N-S for flow near/far from boundary layer.

Conjecture: flow has $\partial_y P \approx 0$, $\partial_x P \approx 0$ also.

Navier-Stokes reduce to: $(\vec{v}\cdot\nabla)\vec{v} = \nu \nabla^2 \vec{v}$

Use incompressibility: $\nabla\cdot\vec{v} = 0 \rightarrow v_i = \varepsilon_{ij}\partial_j \psi$ $\begin{bmatrix} v_x = \partial_y \psi \\ v_y = -\partial_x \psi \end{bmatrix}$

Conjecture 2: $\partial_x \ll \partial_y$ $\quad$ [Note: $\partial_x \sim 1/x$, $\partial_y \sim 1/\delta_b$] $\rightarrow \delta_b \ll x$.

$[|v_y| \ll |v_x|]$

Important N-S: $v_x \partial_x v_x + v_y \partial_y v_x \approx \nu \partial_y^2 v_x$

$$\underbrace{\partial_y\psi\, \partial_x\partial_y\psi - \partial_y^2\psi\, \partial_x\psi}_{\text{contribute at same order}} = \nu\, \partial_y^3 \psi \longrightarrow \text{boundary layer equation.}$$

To solve this nonlinear PDE, first reduce to dim-less variables.

$$\left.\begin{aligned} [\psi] &= [L]^2/[T] \\ [\nu] &= [L]^2/[T] \end{aligned}\right\}$$

Dimensionless vars:

$$\psi = \nu\, \Psi$$

$[v_0] = [L]/[T]$

$[x] = [y] = [L]$ (x has units of length)

$$\left.\begin{aligned} x &= \frac{\nu}{v_0} X \\ y &= \frac{\nu}{v_0} Y \end{aligned}\right\}$$

We're going to solve for $\Psi(X,Y)$: $v_x = v_0\,\partial_Y\Psi$
$v_y = -v_0\,\partial_X\Psi$

and: $\partial_Y\Psi\,\partial_X\partial_Y\Psi - \partial_X\Psi\,\partial_Y^2\Psi = \partial_Y^3\Psi$

Conjecture 3: similarity ansatz:

$$\Psi = X^\alpha f\big(\underbrace{\tfrac{Y}{X^\beta}}_{\text{Define as } \xi}\big) \rightsquigarrow f\left(\tfrac{Y}{\delta_b}\right)$$

For now, α & β are undetermined.

Ansatz won't be consistent for general α:

$$\partial_Y^3\Psi = X^{\alpha-3\beta} f'''(\xi)$$

$$\partial_X\Psi\,\partial_Y^2\Psi = \Big[\alpha X^{\alpha-1} f(\xi) - \underbrace{\beta\frac{Y}{X^{\beta+1}} X^\alpha f'(\xi)}_{\to\ \beta\xi X^{\alpha-1} f'}\Big] X^{\alpha-2\beta} f''(\xi)$$

only compatible w/ sim: $\underline{\alpha - 3\beta = 2\alpha - 2\beta - 1}$ → Solve $\alpha = 1-\beta$.

Fix β by noting $\frac{v_x}{v_0} \to 1$ as $Y\to\infty$, or

$$\partial_Y\Psi\big|_{Y=\infty} = 1 = \underline{X^{\alpha-\beta} f'(\infty)}$$

↳ $f(\xi) \approx \xi$ at large ξ.

So $\alpha = \beta = \tfrac{1}{2}$. → $\Psi \sim \sqrt{X}\, f\left(\frac{Y}{\sqrt{X}}\right)$

Solve for f:

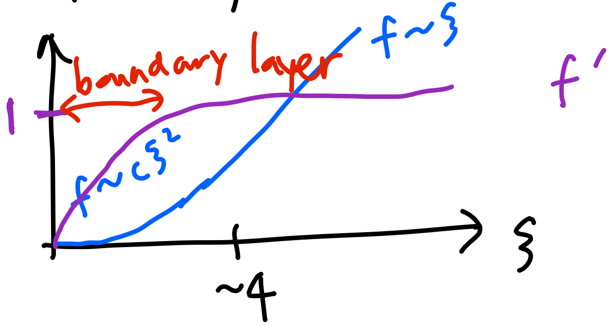
$$\partial_Y^3\Psi = \frac{1}{X} f'''(\xi) = X^{\alpha-\beta} f'\left((\alpha-\beta)X^{\alpha-\beta-1} f' - \beta\xi X^{\alpha-\beta-1} f''\right) + \frac{1}{2} X^{2\alpha-2\beta-1}(\xi f' - f) f''$$

Deduce: $\boxed{f'''(\xi) = -\tfrac{1}{2} f f''(\xi)}$ Blasius' equation (1907)

Boundary conditions: $\frac{v_x}{v_0} = \partial_Y\Psi = \begin{cases} 1 & \text{at } Y=\infty \\ 0 & \text{at } Y=0 \end{cases}$

$$\frac{v_y}{v_0} = -\partial_X\Psi = \frac{1}{2\sqrt{X}}\left[f - \xi f'\right] \to 0 \text{ at } \xi = 0. \rightsquigarrow f(0) = 0.$$

Numerically solve!



When are approximations accurate?

$$\underbrace{\partial_y \Psi}_{\sim v_x} \gg \underbrace{\partial_x \Psi}_{\sim v_y} \qquad \text{or:} \qquad f'(\xi) \gg \frac{1}{2\sqrt{X}}\left[f - \xi f'\right]$$

This is accurate if $X \gg 1$, or

$X = x\frac{v_0}{\nu} \gg 1$ then $x^2 \gg \frac{\nu x}{v_0} = \delta_b^2$, or $x \gg \delta_b$

Approx fails close to onset of boundary layer at $x=0$.

Need to check that pressure gradients are small.

Already used x-NS to find sol'n...

Check approx by looking at y-NS:

$$\underbrace{v_y \partial_y v_y + v_x \partial_x v_y}_{\frac{v_0}{x} v_y} + \frac{\partial_y P}{\rho} \simeq \underbrace{\nu \partial_y^2 v_y}_{\frac{\nu}{\delta_b^2} v_y}$$

$$\frac{v_0}{x} v_y \sim \frac{\nu}{\delta_b^2} v_y$$

using method of dominant balance.

$$\left.\frac{\Delta P}{\rho}\right|_{\text{across boundary layer}} \lesssim \left[\frac{\nu}{\delta_b^2} v_y\right] \cdot \delta_b \sim \frac{\delta_b}{x} v_0 v_y \sim \frac{\delta_b}{x} v_0 \cdot v_0 \frac{\delta_b}{x}$$

So $\frac{\Delta P}{\rho} \sim \left(\frac{\delta_b}{x}\right)^2 v_0^2$ and $\frac{\partial_x P}{\rho} \sim \frac{v_0^2}{x}\left(\frac{\delta_b}{x}\right)^2 \sim \underbrace{(v_x \partial_x v_x)}_{\text{leading order term in x-NS}} \cdot \left(\frac{\delta_b}{x}\right)^2$

small correction!

What is the drag force on plate (per unit width!)

$$F_x = \underset{\text{top + bottom}}{2}\int_0^{L \,\leftarrow\, \text{length of plate}} dx\,(-\tau_{yx}) = 2\eta\int_0^L dx\,\partial_y v_x = 2\eta\int_0^L dx\,\partial_y^2\psi$$

$$\partial_y^2\psi \;\downarrow\; \left(\frac{v_0}{\nu}\partial_Y\right)^2 \nu\Psi$$

$$= 2\eta\,\frac{v_0^2}{\nu}\int_0^L d\!\left(\frac{\nu}{v_0}X\right)\frac{f''(\xi)}{\sqrt{X}}\Bigg|_{\xi\to 0}$$

$$= 2\eta v_0\int_0^{v_0L/\nu}\frac{dX}{\sqrt{X}}\,f''(0) = 4\eta v_0 f''(0)\cdot\sqrt{Lv_0/\nu}$$

$$= 4f''(0)\cdot\rho\sqrt{\nu v_0^3 L}$$

Our calculation valid if $L \gg \sqrt{\frac{\nu L}{v_0}}$

↳ or $\sqrt{\frac{v_0 L}{\nu}} = R \gg 1$

for approx boundary layer flow.

$$F_{x,tot} = \underset{\text{width}}{w}\cdot 4\rho v_0^2 L f''(0)\cdot R^{-1/2} = \frac{\rho v_0^2}{2}\underbrace{A}_{2Lw}\cdot C_D$$

↖ drag coeff.

plate at $R \gg 1$: $C_D \approx 4f''(0)R^{-1/2} \approx 1.3R^{-1/2}$.

Compare to creep flow (Lec 13):

$$F_{x,tot} = 6\pi\rho\nu R v_0 = \frac{\rho v_0^2}{2}A\cdot\boxed{\frac{3}{R}}$$

→ creep flow: $C_D \sim R^{-1}$