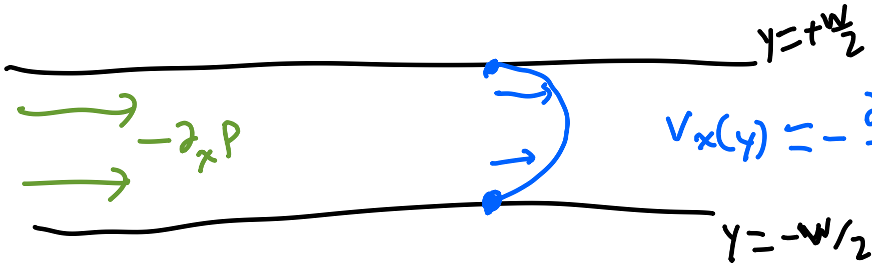
PHYS 7810
Hydrodynamics
Spring 2024

Lecture 15
Instabilities of viscous flows

March 5

Recall: Lec 13, Poiseuille flow thru 2d pipe/channel:



$$v_x(y) = -\frac{\partial_x P}{2\eta}\left(\frac{w^2}{4} - y^2\right) = v_{max}\left(1-\left(\frac{2y}{w}\right)^2\right)$$

Valid solution for any value of Reynolds number:

$$R = \frac{v_{max} w \rho}{\eta} = \frac{v_{max} w}{\nu}$$

In Nature, we'll only find this flow if it is stable:

Let's consider
$$\vec{v} = \vec{v}_0 + \varepsilon \vec{v}_1 + \dots$$
$$P = P_0 + \varepsilon P_1 + \dots$$

unperturbed/Poiseuille

infinitesimally small at $t=0$.

Question: if we use perturbed init. cond. for Navier-Stokes, will $\vec{v}(t\to\infty) \approx \vec{v}_0$?

Assume: that flow remains incompressible $\nabla \cdot \vec{v} = 0$.

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \frac{1}{\rho}\nabla P = \nu \nabla^2 \vec{v}$$

$$\hookrightarrow \partial_t \vec{v}_0 + (\vec{v}_0 \cdot \nabla)\vec{v}_0 + \frac{1}{\rho}\nabla P_0 = \nu \nabla^2 \vec{v}_0 \quad \checkmark$$

$$+ \varepsilon\left[\partial_t \vec{v}_1 + (\vec{v}_1 \cdot \nabla)\vec{v}_0 + (\vec{v}_0 \cdot \nabla)\vec{v}_1 + \frac{1}{\rho}\nabla P_1\right] = \varepsilon\left[\nu \nabla^2 \vec{v}_1\right] + \cdots$$

By construction, $\vec{v}_0$ & P_0 solve above equation at $\mathcal{O}(\varepsilon^0)$

so focus on $\mathcal{O}(\varepsilon)$ terms.

Our background $\vec{v}_0$ is translation invariant in x & t, so

look for $\vec{v}_1 \sim \vec{v}_1(y, k, \omega) e^{ikx - i\omega t}$

The background solution $\vec{v}_0 = U(y)\,\hat{x}$, so

$$-i\omega \vec{v}_1 + \left[v_{1y} \partial_y U\right]\hat{x} + ikU\vec{v}_1 + \begin{pmatrix} ik \\ \partial_y \end{pmatrix}\frac{1}{\rho}P_1 = \nu(\partial_y^2 - k^2)\vec{v}_1$$

Use incompressibility $\nabla \cdot \vec{v}_1 = 0$ to define stream function:

$$v_{1y} = -\partial_x \psi_1 \qquad v_{1x} = \partial_y \psi_1$$

x-comp: $$\frac{P_1}{\rho} = \frac{1}{ik}\left[\nu(\partial_y^2 - k^2)\partial_y \psi_1 + i\omega \partial_y \psi_1 + ik\psi_1 U' - ikU\partial_y \psi_1\right]$$

$$-i\omega(-ik\psi_1) + ikU(-ik\psi_1) + \partial_y\left(\frac{P_1}{\rho}\right) = \nu(\partial_y^2 - k^2)(-ik\psi_1)$$

$$-i\omega\underbrace{(\psi_1'' - k^2\psi_1)}_{B} = \underbrace{-ikU(\psi_1'' - k^2\psi_1) + ikU''\psi_1 + \nu(\psi_1'''' - 2k^2\psi_1'' + k^4\psi_1)}_{A}$$

This is a generalized eigenvalue equation:

$$A\vec{x} = \omega \cdot B\vec{x}$$

The generalized eigenvalues ω are generically complex:

$$\omega = \underset{\text{real}}{\omega' + i\omega''}$$

and $\vec{v}_1 \sim \vartheta_1(y)\, e^{ikx}\, e^{-i\omega' t}\, \boxed{e^{\omega'' t}} \rightarrow$ instability if $\omega'' > 0$.

NB: instability due to boundary conditions, not failure of thermalize

Good practice: work in dimensionless units...

$$U(y) = v_{max}\left(1 - \left(\frac{2y}{w}\right)^2\right) \longrightarrow \tilde{y} = \frac{2y}{w}$$

$$-i\omega\underbrace{\left[\left(\frac{2}{w}\right)^2 \partial_{\tilde{y}}^2 - k^2\right]}\psi_1 = -ikv_{max}(1-\tilde{y}^2)\left(\left(\frac{2}{w}\right)^2\partial_{\tilde{y}}^2 - k^2\right)\psi_1 - 2ikv_{max}\left(\frac{2}{w}\right)^2\psi_1 + \nu\left[\left(\frac{2}{w}\right)^2\partial_{\tilde{y}}^2 - k^2\right]^2\psi_1$$

$$\tilde{k}^2\left(\frac{2}{w}\right)^2 = k^2 \qquad \tilde{\omega} = \frac{\omega}{v_{max}}\frac{w}{2}$$

Reynolds #: $R = \dfrac{v_{max}\, w}{2\nu}$

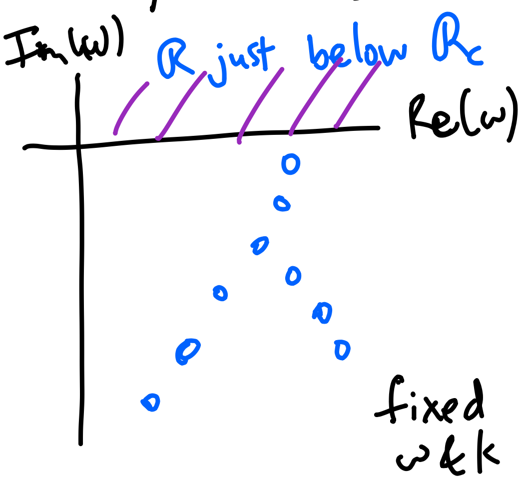
dropping tildes:

Orr-Sommerfeld equation

$$-i\omega(\partial_y^2 - k^2)\psi_1 = -ik(1-y^2)(\partial_y^2 - k^2)\psi_1 - 2ik\psi_1 + \frac{1}{R}(\partial_y^2 - k^2)^2\psi_1$$

Solve this problem numerically: [1972: Orszag used spectral methods]

$\rightarrow$ $y \in [-1, 1]$, we expect a discrete e-val spectrum



Once $R \geq R_c$, Poiseuille flow unstable
b/c "realistic" expt will have some initial condition for unstable k.

What's endpoint of instability?

turbulence! = chaotic flow, have "multi-scale" structure/ vortex dynamics (lec 16)

in contrast to... laminar flow (not turbulent).

Typically, turbulence onsets at $R \sim 10^4$

Example / estimate: everyday flows.

Kinematic viscosity of air $\nu \sim 10^{-5}\ m^2/s$

water $\nu \sim 10^{-6}\ m^2/s$

Human scale fluid flow is turbulent!

$v_{typ} \sim 1\ m/s \quad L \sim 1\ m \rightarrow R \sim 10^5$ (air) $\quad 10^6$ (water)

Linear (in)stability analysis predicted $R_c \approx 5772$.

"puffs" = traveling wave/ t-inhomogeneous at $t \to \infty$ ...

huh?

Linear stability doesn't tell the whole story!

An apparent detour... $R = \underbrace{R_c + \delta}_{\text{slightly unstable}}$ (just above instability!)

($\mathrm{Im}(\omega)$ close to 0)

expect that 1 "unstable" mode + many stable modes...

$$\psi(t) \text{ "="} \Big[A(t)\, \psi_I(y) + \sum_n B_n(t)\, \psi_n(y) \Big] e^{ikx}$$

most unstable (only)

stable

Only the most unstable mode matters? (Landau)

$$\frac{d}{dt}|A|^2 = 2\underbrace{\omega''}_{\mathrm{Im}(\omega)} |A|^2 - \underbrace{\alpha |A|^4 - \beta |A|^6}_{\text{exact computation not realistic...}} + \cdots$$

$|\omega''| \ll -\mathrm{Im}(\omega_{rest})$

just above onset of instability, $\omega'' \approx k(R - R_c) \approx k \cdot \delta$

If $\alpha > 0$ (simplest guess...): for small δ

$\frac{d}{dt}|A|^2 = 0$ when $2k\delta |A|^2 \approx \alpha |A|^4$

or $|A| \approx \sqrt{\frac{2k\delta}{\alpha}}$.

Scaling $|A| \sim \delta^{1/2}$
"mean-field" for continuous transitions.

If $\alpha < 0$ [but $\beta > 0$]:

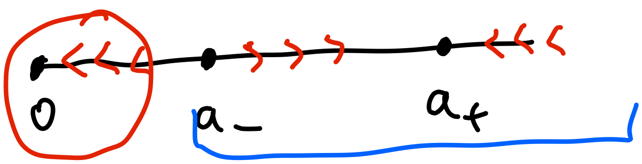
$$\frac{d}{dt}|A|^2 = 0 = 2\delta k |A|^2 + |\alpha||A|^4 - \beta |A|^6$$

$\hookrightarrow$ $|A| = 0$ or $\beta |A|^4 - |\alpha| |A|^2 - 2\delta k$

$$\text{or} \quad |A|^2 = \frac{|\alpha| \pm \sqrt{|\alpha|^2 + 8\beta k \cdot \delta}}{2\beta} = a_{\pm}^2$$

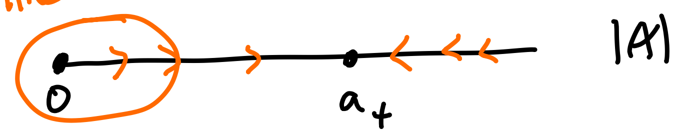
Dynamics of $A(t)$?

$\delta < 0$



linear ~~stability~~ "metastability"

linear inst. $\delta > 0$



"Puff-like" dynamics of Poiseuille for $R \leq 5772$ is understood as "metastability" for $R \gtrsim 3000$.