PHYS 7810
Hydrodynamics
Spring 2024

Lecture 16
Phenomenology of turbulent flow

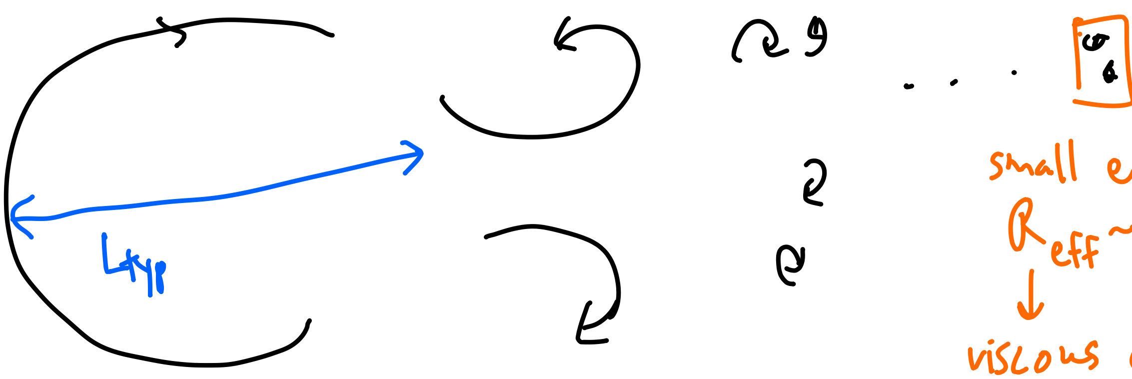
March 7

Recall: high Reynolds number flow $\mathcal{R} \gg 1$:

$$\mathcal{R} = \frac{v_{typ} L_{typ}}{\nu},$$

saw an instability from smooth / lamiar flow to

turbulent flow = chaotic, multi-scale flow



small enough lengths $\mathcal{R}_{eff} \sim 1$

$\downarrow$

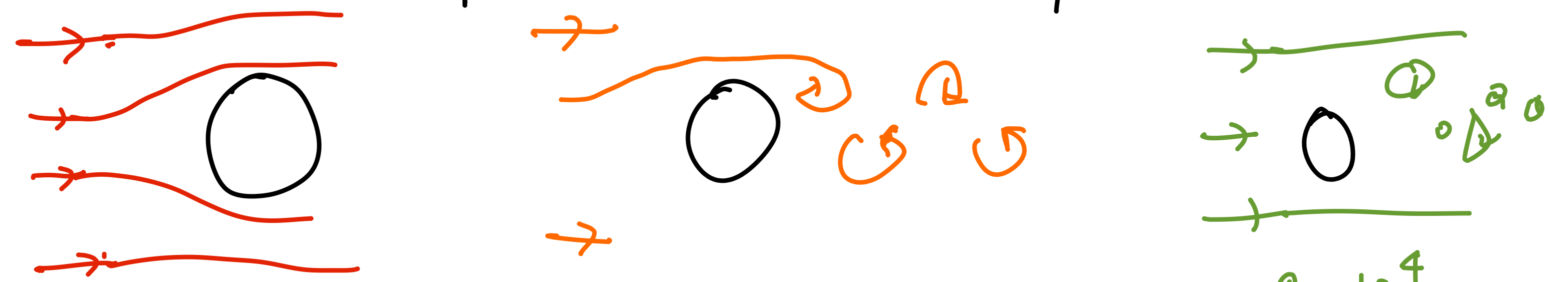
viscous effects dominate

vortex / eddy splits into smaller & smaller eddies. . .

direct cascade = inject energy @ large length scales, energy flows to small length scales.

We don't have rigorous theory for effective description (non-chaotic) of turbulence. $\Longrightarrow$ cartoons!

Illustrative example: wake behind cylinder



$R \sim 1$ laminar

$R \sim 100$ not turbulent, nonlinear!

$R \sim 10^4$ turbulent.

↓

Laminar regime: can use boundary layer theory (lec 14):

$$v_x \partial_x v_x + v_y \partial_y v_x = \nu \partial_y^2 v_x \qquad [\partial_x \ll \partial_y, \ \delta v_x \gg \delta v_y]$$

Far from cylinder: $v_x \simeq v_0 + \underbrace{\delta v_x}_{\text{correction}}$

v_0: const. far from cylinder

$$v_0 \partial_x \delta v_x + \cancel{\delta v_y \partial_y \delta v_x} = \nu \partial_y^2 \delta v_x$$

Explicit solution: $$\delta v_x = \tilde{C} \frac{e^{-y^2/4(\nu x/v_0)}}{\sqrt{4\pi \frac{\nu}{v_0} x}}$$

$\tilde{C}$: integration const. TBD.

$x = 0$: cylinder

$x \gg R$ (radius): $\underbrace{\text{deep in wake}}_{\text{approx. valid.}}$

Estimate drag force/unit length:

$$F_x = \frac{d}{dt}\left[x\text{-momentum carried by fluid}\right] = \int_{-\infty}^{\infty} dy \left[T_{xx}(x \to -\infty) - T_{xx}(x \to +\infty)\right]$$

$$= \rho \int_{-\infty}^{\infty} dy \left[v_0^2 - (v_0 + \delta v)^2\right] \approx 2\rho v_0 \int_{-\infty}^{\infty} dy \, \delta v_x(y) = 2\rho v_0 \tilde{C}$$

Re-introduce drag coefficient (lec 14): C_D

$$C_D \sim \frac{F_x}{\frac{1}{2}\rho v_0^2 \cdot R} \quad \text{so} \quad C_D \sim \frac{\tilde{C}}{R v_0}$$

R = cylinder radius

Usually, we expect at $\mathcal{R} \lesssim 1$ (laminar): $C_D \sim \frac{1}{\mathcal{R}} \times \underbrace{\frac{1}{\log \frac{1}{\mathcal{R}}}}_{\text{cylinder (Stokes' Paradox)}}$

$$F_x \sim \frac{1}{2}\rho v_0^2 R \cdot \frac{1}{\mathcal{R}}$$

$$F_x \sim \frac{1}{2}\rho v_0 \nu \times \frac{1}{\log \frac{\nu}{R v_0}}$$

What happens for turbulent flow?

Weak turbulence approximation: $v_i = \bar{v}_i(x) + \delta v_i(t,x)$

big background

small pert.

statistically average over t-dep.

Expand Navier-Stokes in δv:

$$v_j \partial_j v_i + \frac{\partial_i P}{\rho} = \nu \partial_j \partial_j v_i$$

plug in $\bar{v} + \delta v$, avg over statistical fluctuations... (take $\overline{\delta v_i} = 0$)

$$\bar{v}_j \partial_j \bar{v}_i + \frac{\partial_i \bar{P}}{\rho} + \partial_j\left(\overline{\delta v_j \delta v_i}\right) = \nu \partial_j \partial_j \bar{v}_i$$

Note incompressible $\partial_j \delta v_j = 0$.

Problem: equation for $\bar{v}$ depends on $\overline{\delta v^2}$, $\bar{v}$ depend on turbulent δv...

Strategy #1: EOM for $\overline{\delta v_j \delta v_i}$?

$$\overline{\delta v_k \left[v_j \partial_j v_i + \frac{\partial_i P}{\rho}\right]} = \overline{\delta v_k \nu \partial_j \partial_j v_i} + (i \leftrightarrow k)$$

$$\overline{\delta v_k \delta v_j}\,\partial_j \bar{v}_i + \overline{\delta v_i \delta v_j}\,\partial_j \bar{v}_k \quad \bar{v}_k \partial_j \overline{\delta v_j \delta v_i} + \bar{v}_i \partial_j \overline{\delta v_j \delta v_k}$$

$$+ \overline{\delta v_k \frac{\partial_i \delta P}{\rho}} + \overline{\delta v_i \frac{\partial_k \delta P}{\rho}} + \partial_j \left(\overline{\delta v_j \delta v_i \delta v_k}\right) = \nu\,[\cdots]$$

Solve for $\bar{v}$, $\overline{\delta v^2}$, $\overline{\delta v^3}$, ... until it's too annoying.

Reminiscent to kinetic theory (BBGKY) (lec 21?)

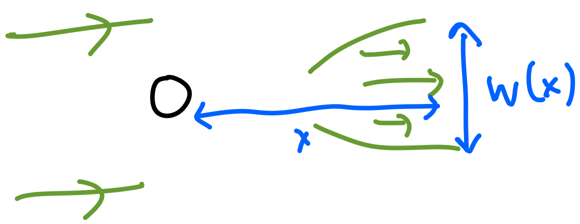
Strategy #2: Phenomenology: $\overline{\delta v_i \delta v_j} = -\nu_t (\partial_i \bar{v}_j + \partial_j \bar{v}_i)$

ν_t "turbulent viscosity"

$\nu_t \sim$ [flow size] $\times$ [velocity scale]

Assume $\nu_t \gg \nu$ ... encoding multi-scale eddy dynamics into ν_t.

Return to wake:



let $\delta v_x(y=0, x) = \delta v_0(x)$

Then $\nu_t \sim \delta v_0(x) \cdot w(x)$.

$$v_0(x)\,\partial_x \delta v_x = \nu_t(x)\,\partial_y^2\, \delta v_x \longrightarrow \delta v_x \sim \tilde{C}(x)\, \frac{e^{-y^2/4(\nu_t x/v_0)}}{\sqrt{4\pi \nu_t \frac{x}{v_0}}}$$

By calculating drag force:

$$F_x = \int dy\, \rho\left[T_{xx}(x\to -\infty) - T_{xx}(x \to +\infty)\right] \approx \rho v_0 \tilde{C} \sim \rho v_0 \cdot \underbrace{w(x)\,\delta v_0(x)}_{}$$

constant! $\rightarrow \nu_t$

Estimate by dimensional analysis: $\nu_t \sim v_0 R \cdot C_D$

Comparing width of wake in laminar vs. turbulent

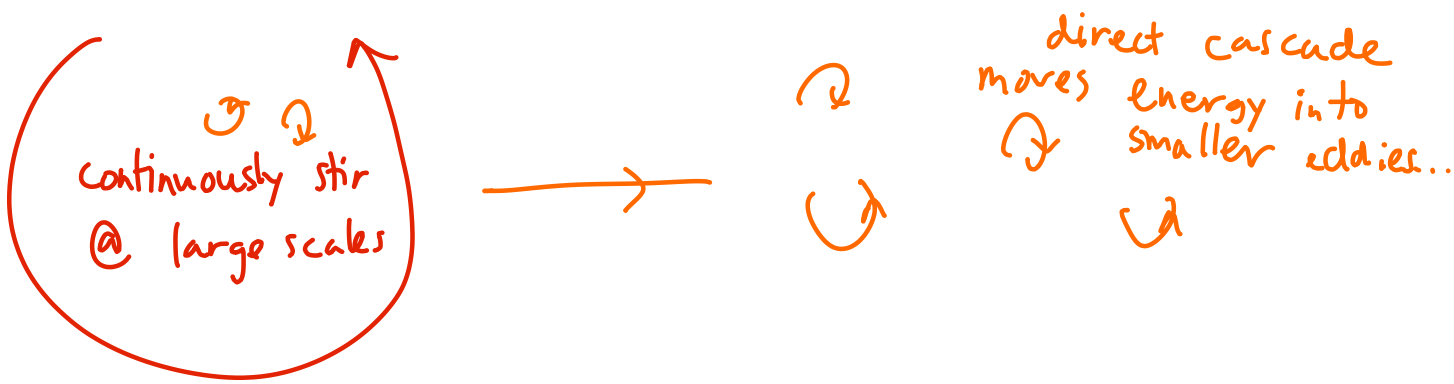
$$w \sim \sqrt{\frac{\nu x}{v_0}} \qquad w \sim \sqrt{\frac{\nu_t x}{v_0}} \sim \sqrt{C_D R x}$$

$$\sim w_{lam} \sqrt{C_D R}$$

Since $R \gg 1$, turbulent wake much larger.

Entrainment = turbulent wake "sucks in" ambient fluid (cf HW4)

Kolmogorov (1941)'s scaling theory for isotropic, driven turbulence



Statistical steady state = injected energy @ large scales → "flow" in k-space from small k → large k.

Let $u(k) = \int d^3\vec{q}\ \delta(|\vec{q}|-k)\ \frac{\rho}{2} \langle \vec{v}(-\vec{q})\, \vec{v}(\vec{q}) \rangle \sim \frac{k^2}{2} \rho\, |\delta \vec{v}(k)|^2_{typ}$

Dimensional analysis:

$$[u(k)] = \frac{1}{[L]^2} \cdot \frac{[M]}{[L]^3} \left[\int d^3x\ e^{-ikr} v \right]^2 = \frac{1}{[L]^2} \frac{[M]}{[L]^3} [L]^6 \frac{[L]^2}{[T]^2} = \frac{[M][L]^3}{[T]^2}.$$

Postulate: $\frac{u(k)}{M} = \varepsilon^{\alpha} k^{\beta}$ — fix const. (α, β)

M ← total mass

$\hookrightarrow \varepsilon = \frac{1}{M} \left(\frac{dE}{dt} \right)_{stirring}$ →

$$\cancel{\frac{\partial u(k)}{\partial t}} + \frac{\partial}{\partial k} \varepsilon = 0$$

$$[\varepsilon] = \frac{1}{[M]} \frac{[M][L]^2}{[T]^{2+1}} = \frac{[L]^2}{[T]^3}$$

Kolmogorov scaling (≈ observed)

Combine: $\boxed{\frac{u(k)}{M} \sim \varepsilon^{2/3} k^{-5/3}} \rightsquigarrow \langle |\delta v(k)|^2 \rangle \sim \frac{u(k)}{k^2} \sim k^{-11/3}$

Over what range of length scales do eddies exist in a turbulent flows?

Define effective Reynolds number at scale k:

$$R(k) \sim \frac{k^{-1} v_{typ}(k)}{\nu} \leftarrow \text{NOT Fourier trans}$$

By dimensional analysis, $v_{typ}(k) \sim \left(\frac{\varepsilon}{k}\right)^{1/3}$.

So $R(k) \sim \frac{\varepsilon^{1/3}}{\nu} k^{-4/3}$.

Turbulent flow stop at k_{max} where $R(k_{max}) \sim 1$

$$\downarrow$$

$$k_{max} \sim \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$$

Initial flow has Reynolds R_0 ...

$$\left.\frac{R_0}{1} \sim \frac{k_{min}^{-4/3}}{k_{max}^{-4/3}}\right\} \longrightarrow L_{smallest} \sim L_{largest} \cdot R^{-3/4}$$