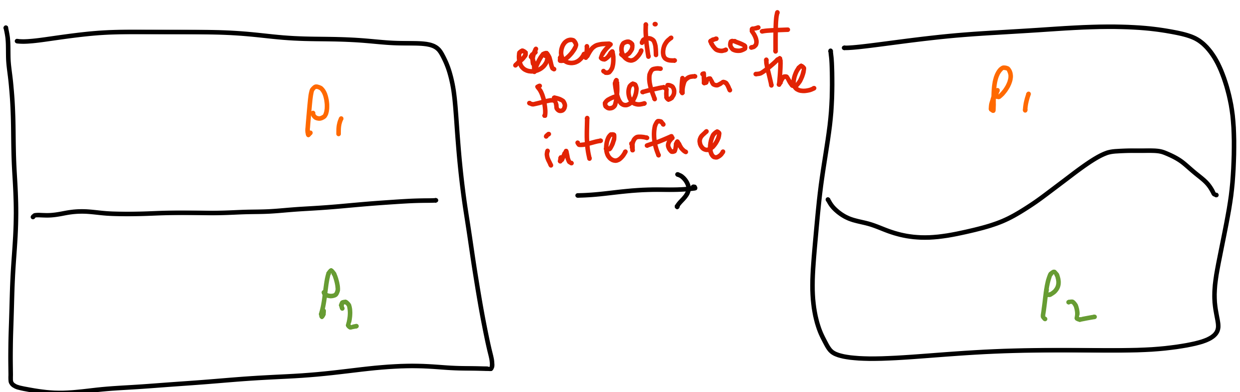
PHYS 7810
Hydrodynamics
Spring 2024

Lecture 17
Surface tension and gravity waves

March 12

Today: interface between 2 fluids $\rightarrow$ surface tension

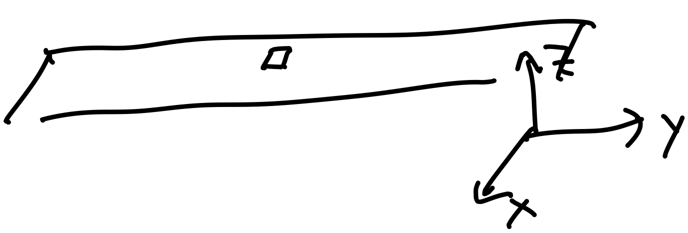


Postulate: work needed for this deformation:

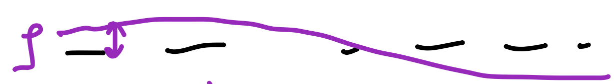
$$dW = \alpha \cdot dA$$

$\alpha > 0$: coefficient of surface tension.

What's dA? Assume interface $\approx$ flat plane at $z = 0$:

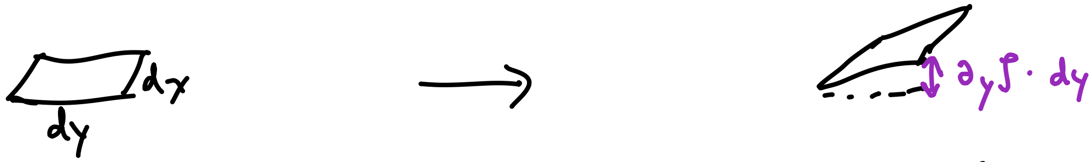


$\rightarrow$



$z = \zeta(x, y)$ parameterizes interface: $|\partial_x \zeta| \ll 1$

For small ζ, what's area of this interface?



Change in area: calculate by looking at induced metric:

$$ds^2 = dx^2 + dy^2 + dz^2 \longmapsto dx^2 + dy^2 + (\partial_x \zeta\, dx + \partial_y \zeta\, dy)^2$$

$$= (dx \;\; dy) \underbrace{\begin{pmatrix} 1+(\partial_x\zeta)^2 & \partial_x\zeta\,\partial_y\zeta \\ \partial_x\zeta\,\partial_y\zeta & 1+(\partial_y\zeta)^2 \end{pmatrix}}_{g} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

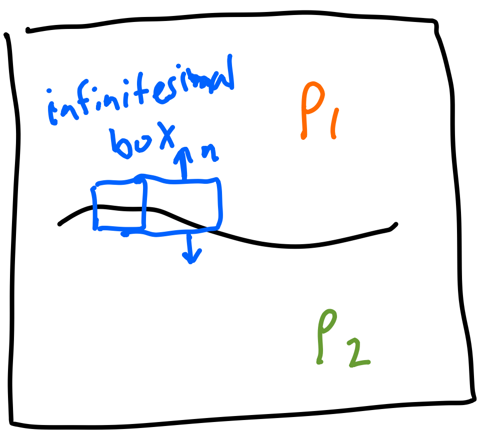
$$d(\text{Area}) = dx \cdot dy \cdot \sqrt{\det(g)}$$
$$= dx\,dy\,\sqrt{1+(\partial_x\zeta)^2+(\partial_y\zeta)^2}$$

$$U = \alpha \int dx\,dy\,\sqrt{1+(\partial_x\zeta)^2+(\partial_y\zeta)^2} \approx \alpha\,(\text{const.}) + \frac{\alpha}{2}\int dx\,dy\left[(\partial_x\zeta)^2+(\partial_y\zeta)^2\right].$$

Force per unit area acting on interface:

$$f_z = -\frac{\delta U}{\delta \zeta} = -\alpha\left[-\partial_x^2\zeta - \partial_y^2\zeta\right] = \alpha \nabla_\perp^2 \zeta$$

How does this f_z modify BCs?



uses divergence thm

$$\cancel{\frac{d}{dt}\rho v_z} + \oint_{\text{box}} (n_j \tau_{jz}) = f_z$$

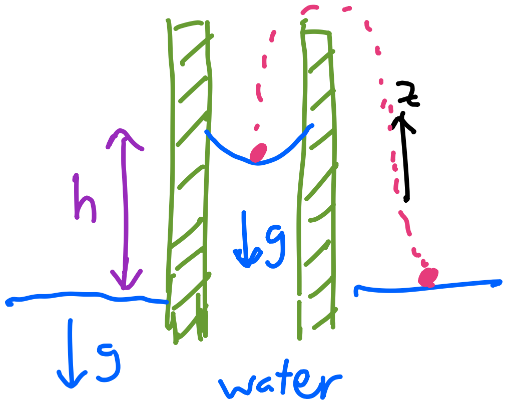
$$= \underbrace{\tau_{zz}(z>0) - \tau_{zz}(z<0) = \alpha \nabla_\perp^2 \zeta}$$

$\downarrow$ new boundary condition for flows w/ surface tension.

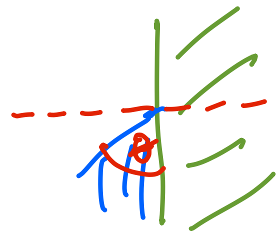
dominant contribution

In general: $\tau_{zz} = P - \eta\left(2\partial_z v_z - \frac{2}{3}\nabla\cdot v\right) - \zeta_{\text{bulk}}\nabla\cdot v$

Example 1: capillary action in plants



How can water rise above bath?



$\theta < \frac{\pi}{2}$: hydrophilic interaction w/ wall.

Fluid is not moving ($\vec{v} = \vec{0}$), Navier-Stokes equations:

$$\cancel{\vec{v}\cdot\nabla\vec{v}} + \frac{1}{\rho}\nabla P = \cancel{\nu\nabla^2\vec{v}} - g\hat{z}$$

$\hookrightarrow P = P_0 - \rho g z$ ↙ hydrostatic pressure.

Pressure approximately equal at surface...

$$|\Delta P_{stem}| \approx \rho g(h+\zeta)$$

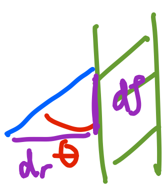
Due to surface tension: $\nabla_\perp^2 \zeta > 0$

$$\Delta P_{stem} = \alpha \nabla_\perp^2 \zeta \approx \rho g h$$

Assume cylindrical stem: $\nabla_\perp^2 \zeta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right) = \frac{\rho g h}{\alpha}$

Show: $\zeta(r) = Cr^2 \longrightarrow \quad 4C = \rho g h / \alpha$

Relate C to θ: if θ close to $\pi/2$

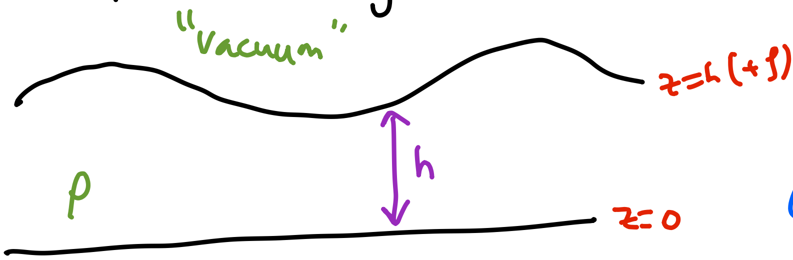


$$\frac{d\zeta}{dr} = 2C\cdot R = 2CR \big/ \tan\theta \approx R\cdot 2C\cos\theta$$

radius of stem

↳ Jurin's Law: $\boxed{h = \frac{2\alpha\cos\theta}{\rho g R}}$

Example 2: gravity waves



Approximate incompressible ($\nabla\cdot\vec{v}=0$)

AND irrotational: $\nabla\times\vec{v}=\vec{0}$.

$\hookrightarrow$ by lec 10: $\vec{v}=-\nabla\Phi$, $\nabla^2\Phi=0$.

For small fluid velocity: Navier-Stokes:

$$\frac{\partial}{\partial t}(-\nabla\Phi)+\frac{1}{\rho}\nabla P\approx -g\hat{z}$$

$$\hookrightarrow \nabla\left[-\frac{\partial\Phi}{\partial t}+\frac{1}{\rho}P+gz\right]=0. \qquad \text{or } -\frac{\partial\Phi}{\partial t}+\frac{1}{\rho}P+gz=\text{const.}$$

At interface: $z=h+\xi$:

$$P_0-P(z\approx h)=\alpha(\partial_x^2+\partial_y^2)\xi$$

$$\alpha(\partial_x^2+\partial_y^2)\xi=-\rho\frac{\partial\Phi}{\partial t}\Big|_{(x,y,z=h,t)}+\rho g\xi(x,y,t)$$

Combine with $\nabla^2\Phi=0$ for $0\le z\le h$.

$\hookrightarrow$ Boundary conditions: at $z=0$, $v_z=0=\partial_z\Phi$

at $z=h$, $v_z=\frac{\partial\xi}{\partial t}$ at $z=h$.

Use (x,y)-translation (& rotational...) to look for solutions:

$$\xi=\xi_0\cdot e^{ikx-i\omega t} \quad\text{and}\quad \Phi=e^{ikx-i\omega t}f(z)$$

Using Laplace: $\nabla^2\Phi=0=(\partial_x^2+\partial_z^2)\Phi=-k^2f+f''$

$\downarrow$

BC at $z=0$: $f(z)=A\cdot\cosh(kz)$

Use linearity to set $A=1$.

Next: BC at $z=h$: $-i\omega\xi_0=-\frac{\partial\Phi}{\partial z}\Big|_{z=h}=-k\sinh(kh)$

$$\text{or}\quad \xi_0=\frac{k}{i\omega}\sinh(kh)$$

Finally: $-\rho \frac{\partial \Phi}{\partial t}\Big|_{z=h} = (-\rho g - \alpha k^2)\,\zeta$

$$+ i\omega \cdot \cosh(kh) = -\left(g + \frac{\alpha}{\rho}k^2\right)\frac{k}{i\omega}\sinh(kh)$$

$$\omega^2 = \left(gk + \frac{\alpha}{\rho}k^3\right)\tanh(kh)$$

$$\underline{\omega = \pm\sqrt{\left(g + \frac{\alpha}{\rho}k^2\right)k\tanh(kh)}}$$

dispersion of gravity waves

Deep water: $h \to \infty$, $\omega = \pm\sqrt{\left(g+\frac{\alpha}{\rho}k^2\right)k}$

Example 3: Tsunami ... over deep ocean

$h \sim 10$ km

wavelength $\lambda \sim 100$ km

$$\underbrace{}_{kh \ll 1}$$

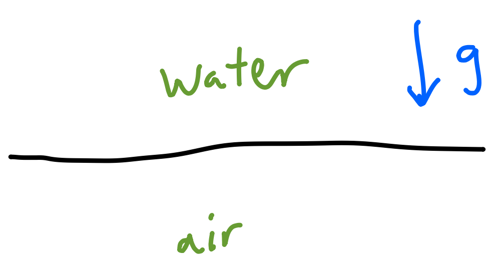
$k \to 0$: $\omega \approx \pm\sqrt{gh}\,k$

Speed $v \sim \sqrt{gh} \sim \sqrt{10 \cdot 10^4}\ \frac{m}{s} \sim 300\ m/s$

time for tsunami to cross Pacific ($L \sim 10^4$ km)

$$t \sim \frac{L}{v} \sim \frac{10^4\ \text{km}}{0.3\ \text{km/s}} \sim 3\times10^4\ \text{s} \sim 10\ \text{hr}$$

Example 4: formation of water droplets...



at this interface, instability due to formations of water droplets..

what's size of water droplet?

Take gravity wave dispersion & take $g \to -g$:

$$\omega = \pm\sqrt{k\left(\frac{\alpha}{\rho}k^2 - g\right)}$$

Instability if $\mathrm{Im}(\omega) > 0$, occurs when $k < k_c = \sqrt{\frac{\rho g}{\alpha}}$

For water: $\rho \sim 10^3\ \mathrm{kg/m^3}$, $g \sim 10\ \mathrm{m/s^2}$, $\alpha \sim 0.07\ \frac{N}{m}$

for water-air interface.

$k_c \sim 400\ m^{-1} \rightsquigarrow l_c = \frac{2\pi}{k_c} \sim 1.5\ cm$.