PHYS 7810
Hydrodynamics
Spring 2024

Lecture 18
Atmospheric fluid dynamics

March 19

Today: Navier-Stokes in rotating reference frame.

In inertial reference frame:

$$\underbrace{\partial_t v_i + v_j \partial_j v_i}_{\frac{d}{dt} v_i} = \underbrace{-\frac{\partial_i P}{\rho} + \nu \partial_j \partial_j v_i}_{\frac{1}{\rho} f_i} \quad \text{and} \quad \partial_i v_i = 0.$$

(convective derivative)

(force per unit volume $= f_i$)

In a non-inertial frame, add "fictitious forces"

Single particle: $x_i(t) = R_{ij}(t)\, X_j(t)$

x_i: non-rotating

X_j: coord in non-inertial frame

$\dot{R}_{ij} = R_{ik}\varepsilon_{klj}\Omega_l$ ← ang. velocity

$$\frac{d^2 x_i}{dt^2} = \frac{d}{dt}\left[\dot{R}_{ij}X_j + R_{ij}\dot{X}_j\right] = \frac{d}{dt}\left[R_{ij}\left[\dot{X}_j + \varepsilon_{jkl}\Omega_k X_l\right]\right]$$

$$= R_{ij}\Big[\ddot{X}_j + \underbrace{2\varepsilon_{jkl}\Omega_k \dot{X}_l + \varepsilon_{jkl}\Omega_k \varepsilon_{lmn}\Omega_m X_n}_{\text{"non inertial force/mass"}}\Big]$$

Coriolis

$$\frac{\vec{f}^{\,NI}}{\rho} = -2\vec{\Omega}\times\vec{v} - \vec{\Omega}\times(\vec{\Omega}\times\vec{x})$$

← centrifugal

So Navier-Stokes becomes:

$$\partial_t \vec v + \vec v \cdot \nabla \vec v = -\frac{\nabla P}{\rho} + \nu \nabla^2 \vec v - 2\vec\Omega \times \vec v - \underbrace{\vec\Omega \times (\vec\Omega \times \vec v)}_{\substack{-|\vec\Omega|^2 \vec x + \vec\Omega(\vec\Omega\cdot\vec x) \\ = -\frac{1}{2}\nabla\left[\Omega^2 x^2 - (\vec\Omega\cdot\vec x)^2\right]}}$$

Define $P' = P - \frac{\rho}{2}\left[\Omega^2 x^2 - (\vec\Omega\cdot\vec x)^2\right]$ then

$$\partial_t \vec v + \vec v \cdot \nabla \vec v = -\frac{\nabla P'}{\rho} + \nu \nabla^2 \vec v - \underline{2\vec\Omega \times \vec v}.$$

Look for dimensionless parameters:

Reynolds number: $R \sim \frac{\text{"}\vec v \cdot \nabla \vec v\text{"}}{\text{"}\nu \nabla^2 \vec v\text{"}} \sim \frac{v_{typ} L_{typ}}{\nu}$

Rossby number: $\Upsilon \sim \frac{\text{"}\vec v \cdot \nabla \vec v\text{"}}{\text{"}2\vec\Omega \times \vec v\text{"}} \sim \frac{v_{typ}}{L_{typ}\Omega}$ } $\Upsilon \ll 1$ then Coriolis dominates

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Example 1: Storms

$v_{typ} \sim 10$ m/s ; $L \sim 1000$ km ; $\nu \sim 10^{-6}$ m^2/s ; $\underbrace{\Omega \sim 10^{-4}\ s^{-1}}$

$\frac{2\pi}{1\ \text{day}}$ = Earth rot. rate

$R \sim 10^{13}$ $\qquad$ $\underbrace{\Upsilon \sim 10^{-1}}_{\text{Coriolis wins}}$

If Coriolis dominates: & static ($\partial_t = 0$) $\approx$ geostrophic flow:

$$\cancel{\partial_t \vec v} + \cancel{\vec v \cdot \nabla \vec v} = -\frac{\nabla P'}{\rho} + \cancel{\nu \nabla^2 \vec v} - 2\vec\Omega \times \vec v$$

or $\quad 2\vec\Omega \times \vec v = -\frac{\nabla P'}{\rho}$

Example 2: storm, again.

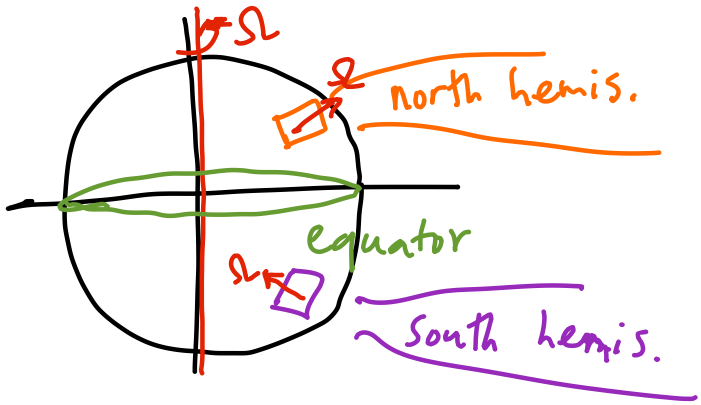
$\rho \sim 1$ kg/m^3

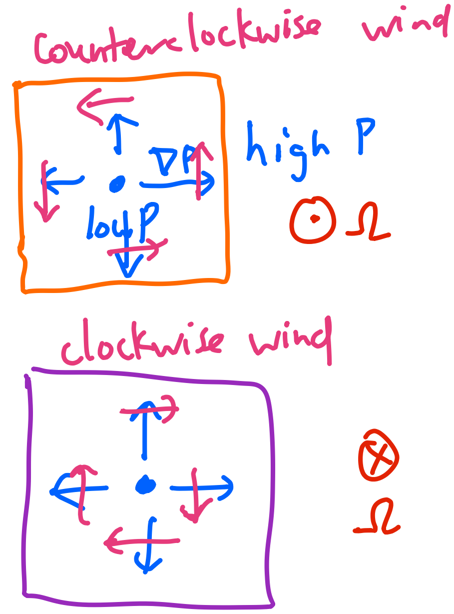
estimate

$$\Delta P' / \rho L_{typ} \sim 2\Omega v_{typ}$$

or $\Delta P' \sim 2$ kPa $\sim 2\%$ of atmospheric pressure.

Which way winds blow?



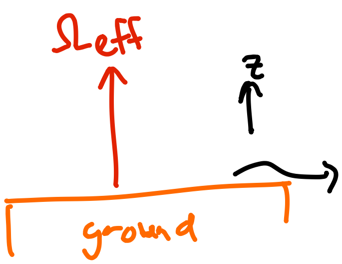


Taylor–Proudman Theorem: for geostrophic flow

$$\vec{0} = -\frac{\nabla P'}{\rho} - 2\vec{\Omega}\times\vec{v} \quad \rightsquigarrow \quad \nabla\times(\vec{\Omega}\times\vec{v}) = 0$$

$$\Downarrow$$

$$0 = (\vec{\Omega}\cdot\nabla)\vec{v}$$



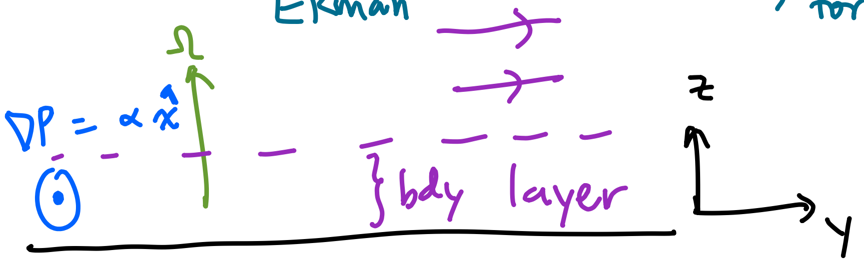
since $v_z = 0$ at $z = 0$, $\Omega_{eff}\,\partial_z \vec{v} = 0$

Deduce flow pattern $\approx$ 2d ; $\vec{v} \approx v_x\hat{x} + v_y\hat{y}$

holds if $Y \ll 1$

Cf lec 14: Ekman boundary layers? e.g. to impose no-slip

↳ for $\approx$ geostrophic flows



far from $z=0$,

$$v_0 = \frac{2\alpha}{\rho\Omega}$$

but (Ekman) bdy layer enforces no-slip...

As before, $P \approx$ const. in our boundary layer

$$0 \approx \nu \nabla^2 \vec{v} - 2\vec{\Omega} \times \vec{v} - \frac{\nabla P'}{\rho} \Big\} \text{ there far from } z \to 0$$

Approx. that $\vec{v} = v_0 \hat{y} + \delta v_y \hat{y} + \delta v_x \hat{x}$

$$\frac{d^2}{dz^2} \delta v_y = \frac{2\Omega}{\nu} \delta v_x \qquad \frac{d^2}{dz^2} \delta v_x = -\frac{2\Omega}{\nu} \delta v_y$$

$$\frac{d^2}{dz^2}(\delta v_y + i\delta v_x) = \frac{2\Omega}{\nu}(-i)(\delta v_y + i\delta v_x)$$

so $\delta v_y + i\delta v_x = A e^{\sqrt{-i2\Omega/\nu}\, z} + B e^{-\sqrt{-i2\Omega/\nu}\, z}$

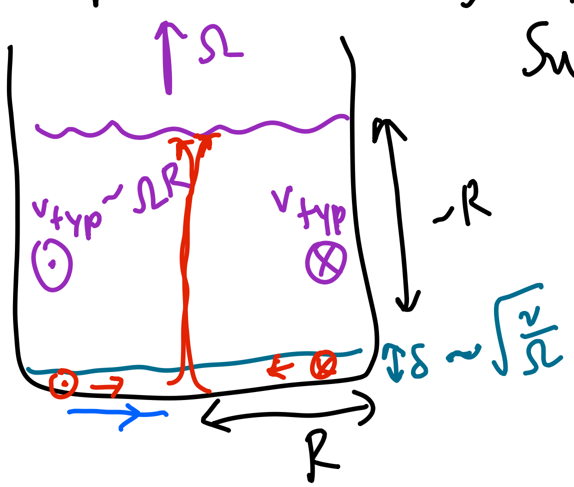
No-slip: $\delta v_x = 0$ and $\delta v_y = -v_0$ at $z = 0$.

Since $\sqrt{-i} = e^{-i\pi/4} = \frac{1-i}{\sqrt{2}} \sim\!\!\to$ $A = 0$ (well-behaved as $z \to \infty$)

$B = -v_0$

Boundary layer thickness: $\delta \sim \sqrt{\frac{\nu}{2\Omega}}$

Example 3: mixing by stirring



Suppose bottom is static

Ekman boundary layer

has radial fluid flow...

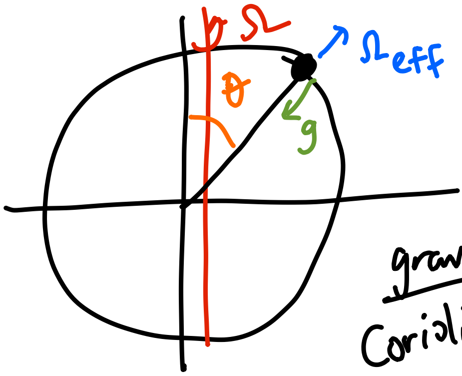
... and mixing of fluid.

Time to mix: $t_{mix} \sim \underbrace{\frac{R}{v_{typ}}}_{\text{time to move thru Ekman layer}} \times \underbrace{\frac{R}{\delta}}_{\text{fraction of fluid in bdy layer}} \sim \frac{1}{\Omega} \frac{R}{\sqrt{\frac{\nu}{\Omega}}} \sim \frac{R}{\sqrt{\nu\Omega}}$

Compare to diffusive mixing time?

$t_{mix} \sim R^2/\nu$, which is much slower.

Return to flow on Earth's surface:



Goal: justify that $\vec{\Omega}_{eff} \sim \hat{r}$

Estimate relative strength of accelerations:

$$\frac{\text{grav}}{\text{Coriolis}} \sim \frac{g}{v_{typ}\,\Omega} \sim \frac{10}{10 \cdot 10^{-4}} \sim 10^4 \gg 1$$

So gravity wins → ≈ hydrostatic equilibrium near surface.

$$\partial_t \vec{v} \approx 0 = -\frac{1}{\rho}\nabla P - g\hat{r} - \vec{\Omega}\times(\vec{\Omega}\times\vec{x})$$

$$= -\frac{1}{\rho}\nabla\underbrace{\left[P + \rho g r - \frac{\Omega^2}{2}\rho\,(\underbrace{x^2+y^2}_{r^2\sin^2\theta})\right]}_{\text{constant.}}$$

$P \approx P_0 - \cancel{\rho g r}$

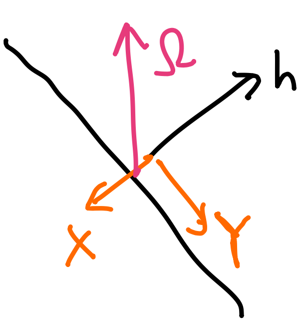
$- \rho g\,(\underbrace{r - \frac{\Omega^2}{2g}R^2\sin^2\theta}_{h})$ Earth's rad.

(effective)
(atmospheric)

What about Coriolis forces?

$$\partial_t \vec{v} \approx 0 \approx -\frac{1}{\rho}\nabla(P + \rho g h) - 2\vec{\Omega}\times\vec{v}$$

Introduce new coordinate system:



(local rectangular coords)

Approx 1: atmospheric thickness $\underline{\Delta h} \ll 1000$ km, so

$$\nabla \cdot \vec{v} = 0 = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \underline{\frac{\partial v_h}{\partial h}} \rightsquigarrow v_h \ll v_x, v_y$$

Approx 2: $\vec{\Omega} = \Omega(\cos\theta\, \hat{h} - \sin\theta\, \hat{y})$

$$\vec{\Omega} \times \vec{v} = \underbrace{\Omega \cos\theta \left[v_x \hat{y} - v_y \hat{x} \right]} - \underbrace{\cancel{\Omega \sin\theta\, v_h \hat{x}}}_{\text{ignore, } v_h \ll v_y} + \underbrace{\cancel{\Omega \sin\theta\, v_x \hat{h}}}_{\text{negligible relative to hydrostatic}}$$

deduce an effective 2d model

$$\Omega_{eff} = \Omega \cdot \cos\theta$$

as claimed, project $\vec{\Omega}$ onto vertical.

In latitude θ_L: $\theta_L = \frac{\pi}{2} - \theta \longrightarrow \Omega_{eff} = \Omega \sin\theta_L$

vanish at equator ($\theta_L = 0$).

Deduce: 2d fluid equation for atmospheric flow:

$$\partial_t v_i + v_j \partial_j v_i = -\frac{1}{\rho} \partial_i \tilde{P} - f \varepsilon_{ij} v_j$$

$i = x, y$ $\tilde{P} = P + \rho g h$ $f = 2\Omega_{eff}$