PHYS 7810
Hydrodynamics
Spring 2024

Lecture 19
Hall viscosity

March 21

Today: Navier-Stokes w/ broken spacetime symmetry

Recall (lec 8&9) EFT for fluids...

0 w/ Galilean boost

$$\mathcal{L} = \pi_\rho \partial_t \rho + \pi_i \partial_t g_i - \partial_i \pi_\rho \left(\rho v_i - \cancel{A} \partial_i (i\pi_\rho + \mu_\rho) \right) - \partial_j \pi_i \left(P\delta_{ji} - T \underbrace{\eta_{jike}}_{\text{viscosity tensor had only 2 ind. coeff (Galilean...)}} \partial_k (i\pi_\ell + \mu_\ell) \right)$$

with broken spacetime symmetries, generally: $A \neq 0$
new viscosities...

What happens if we break time-reversal symmetry.

$g_i \to -g_i$ $\qquad \pi_\rho \to -\pi_\rho + i\mu_\rho$ $\qquad \pi_i \to \pi_i - i\mu_i$

$\mu_i \to -\mu_i$

At ideal fluid level, $\mathcal{L} \not\to \mathcal{L}$

$\to \mathcal{L}_{MSR}$ (still vanish if $\pi_\rho = 0$, $\pi_i = 0$)

Derivative corrections to constitutive relations: e.g. viscosity

$$(\partial_j \pi_i)\, T \eta_{jikl}\, \partial_k (i\pi_l + \mu_l) \to \partial_j(\pi_i - i\mu_i)\, T \eta_{jikl}\, \partial_k \underbrace{(i(\pi_l - i\mu_l) - \mu_l)}_{= i\pi_l}$$

$$= T \eta_{\underbrace{ji}\underbrace{kl}} (\underbrace{\partial_k \pi_l})\, \partial_j(i\pi_i + \mu_i)$$

$$\eta^{rev}_{\underbrace{ji}\,\underbrace{kl}} = \eta_{klji} \qquad = T \eta^{rev}_{jikl} (\partial_j \pi_i)\, \partial_k (i\pi_l + \mu_l)$$

Broken T (time-reversal) allows $\eta_{klji} \neq \eta_{jikl}$

"antisymmetric" $\frac{1}{2}(\eta_{jikl} - \eta_{klji})$ called odd / Hall viscosity

Physical manifestations:

- fluid in background magnetic field
- active fluids

Focus on $d=2$ spatial dimensions. Keep rotation symmetry (this makes $d=2$ special... no odd visc in $d \neq 2$)

think of $\eta_{\underbrace{ji}\underbrace{kl}}$ as 4×4 matrix:

$$\eta_{jikl} = \begin{array}{c} \\ I \\ \sigma^x \\ \sigma^z \\ \varepsilon \end{array} \begin{array}{c} \begin{array}{cccc} I & \sigma^x & \sigma^z & \varepsilon \end{array} \\ \left(\begin{array}{c|cc|c} \rho & 0 & 0 & 0 \\ \hline 0 & \eta & -\eta_H & 0 \\ 0 & \eta_H & \eta & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

rotation symmetry [O(2)...] enforces block diagonal
 - each block is irrep

$\varepsilon \otimes \varepsilon$ part $= 0$ due to angular momentum conservation

If $O(2)$ + broken T: no new viscosity...

Hall viscosity $\rightarrow$ break parity!

$$-\eta_H\left(\sigma^x_{ji}\sigma^z_{kl} - \sigma^z_{ji}\sigma^x_{kl}\right) = \begin{cases} -\eta_H & jikl = xyxx,\ yxxx,\ yyxy,\ yyyx \\ +\eta_H & jikl = xyyy,\ yyxy,\ xxxy,\ xyxx \end{cases}$$

Re-write whole viscous stress tensor:

$$T^{(visc)}_{ji} = -\zeta\delta_{ji}\partial_k v_k - \eta\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}\right]\partial_k v_l - \eta_H\left(\delta_{ik}\epsilon_{jl} + \delta_{il}\epsilon_{jk}\right)\partial_k v_l$$

What does η_H do to flows / Navier-Stokes? (Assume $\partial_i v_i = 0$)

$$\partial_j\Big[\overset{\text{const.}}{\eta_H}\left(\delta_{ik}\epsilon_{jl} + \delta_{il}\epsilon_{jk}\right)\partial_k v_l\Big] = \eta_H\,\partial_i\epsilon_{jl}\partial_j v_l = \eta_H\,\partial_i\underset{\text{vorticity}}{\omega}$$

For incompressible flow:

$$\partial_t v_i + v_j\partial_j v_i + \frac{\partial_i P}{\rho} - \nu\partial_j\partial_j v_i - \frac{\eta_H}{\rho}\partial_i\omega = 0$$

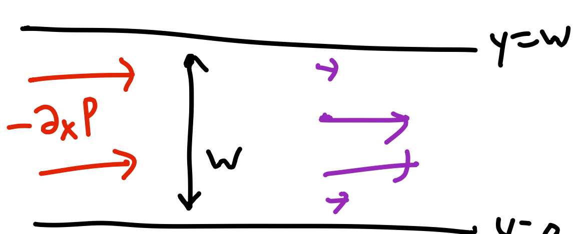
$= \frac{1}{\rho}\partial_i(P - \eta_H\omega)$: η_H "just" leads to pressure offset

Solutions $v_i = \epsilon_{ij}\partial_j\psi$:

$$\partial_t\partial_j\partial_j\psi + \partial_k\left[\epsilon_{jl}\partial_l\psi\,\partial_j\partial_k\psi\right] - \nu\,\partial_i\partial_i\partial_j\partial_j\psi = 0. \qquad (\text{no } \eta_H)$$

Deduce η_H only modifies "boundary conditions" (or P)

Example: Poiseuille / pipe flow (lec 13)



$$v_x(y) = -\frac{\partial_x P}{\eta}\,y(w-y)$$

y-Navier-Stokes: $\partial_y(P-\eta_H\omega)=0$ or $P=\eta_H\omega$

$$\omega = \partial_x v_y - \partial_y v_x = -\frac{\partial_x P}{\eta}(2y-w)$$

Hall viscosity → measure transverse pressure $\Delta P_{trans} = \eta_H\left(-\frac{\partial_x P}{\eta}2w\right)$

Fluid with Hall viscosity has generalized time-reversal symmetry:

time-reversal + parity = symmetry

$P\cdot\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-x\\ y\end{pmatrix}$ takes $\varepsilon_{ij} = -\varepsilon_{ij}$; $\delta_{ij}\to\delta_{ij}$

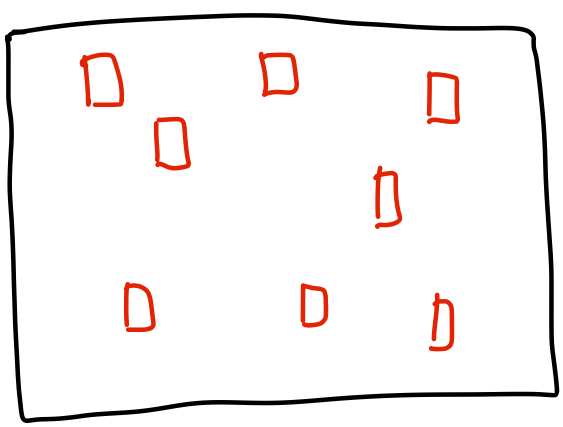
$$\mathcal{L} \supset \partial_j\pi_i\, T\eta_A \underbrace{(\delta_{ik}\varepsilon_{jl}+\delta_{il}\varepsilon_{jk})}\partial_k(i\pi_l+\mu_l)$$

→ $-1\times(\cdots)$ under parity

→ $-1\times(\cdots)$ under T

Most T-broken fluids have some generalized T

We can break rotational symmetry instead...



rotational symmetry O(2) + parity

broken to dihedral D_8/D_4

symmetries remain:

90° rot $(x,y)\to(y,-x)$

parity $(x,y)\to(-x,y)$

Restore T-symmetry.

$$\mathcal{L}\supset T\eta_{jikl}(\partial_j\pi_i)(\partial_k(i\pi_l+\mu_l))\text{, and } \eta_{jikl}=\eta_{klji}$$

$$
\eta_{jik\ell} = \begin{array}{c} I \\ \sigma^x \\ \sigma^z \\ \varepsilon \end{array}
\left(\begin{array}{c|c|c|c}
\zeta & 0 & 0 & 0 \\ \hline
0 & \underline{\eta_x} & 0 & 0 \\ \hline
0 & 0 & \underline{\eta_z} & 0 \\ \hline
0 & 0 & 0 & \boxed{\eta_r}
\end{array}\right)
$$

each block in 1d irrep

rotational viscosity! discrete rotations → no angular momentum.

2 shear viscosities! (cf elastic solids)

Simpler argument:

$\eta_{jik\ell}$ needs to be invariant under $\underbrace{90^\circ \text{ rot}}$ & $\underbrace{\text{parity}}$

$\hookrightarrow R_{ii'} R_{jj'} R_{kk'} R_{\ell\ell'} \eta_{j'i'k'\ell'}$

$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\eta_{yyxx} \to \eta_{(-x)(-x)yy}$ $\eta \to (-1)^{\#\text{ of } x}\eta$

flip $y \leftrightarrow x$ ← only have 0, 2, 4 x

Count independent coeff:

① $\eta_{xxxx} = \eta_{yyyy}$

② $\eta_{xxyy} = \eta_{yyxx}$

$\eta_{xyxy} = \eta_{yxyx}$

$\eta_{xyyx} = \eta_{yxxy}$

$$4\zeta = 2(\eta_{xxxx} + \eta_{xxyy})$$

$$4\eta_z = 2(\eta_{xxxx} - \eta_{xxyy})$$

$$4\eta_x = 2(\eta_{xyxy} + \eta_{xyyx})$$

$$4\eta_r = 2(\eta_{xyxy} - \eta_{xyyx})$$

Anisotropic (non-thermal) fluids can also have unusual ideal fluid constitutive relations!

Consider fluid w/ triangular symmetry (120° rot)

$\rightarrow$ invariant λ_{ijk} (fully-symmetric)

$$\mathcal{L} = \pi_\rho \partial_t \rho + \pi_i \partial_t g_i - \partial_i \pi_\rho J_i - \partial_j \pi_i \tau_{ji}$$ (broken T)

Guess: $J_i = A\mu_i + B\lambda_{ijk}\mu_j\mu_k$

$$\tau_{ji} = C\delta_{ji} + D\mu_j\mu_i + F\lambda_{ijk}\mu_k + G\mu_i\lambda_{jkl}\mu_k\mu_l + H\mu_j\lambda_{ikl}\mu_k\mu_l$$

$$\mathcal{L} \xrightarrow{T} \mathcal{L} - \underbrace{i\,\partial_i\mu_\rho(A\mu_i + B\cdots) - i\,\partial_j\mu_i(C\delta_{ji} + \cdots)}$$

compare...

$= i\,\partial_i S^i$

$\hookrightarrow S^i = Q\mu_i + R\lambda_{ijk}\mu_j\mu_k$

$C = Q$

$A = \dfrac{\partial Q}{\partial \mu_\rho}$ etc...

For systems in thermal eq, try to couple to background fields...

$R =$ const. (new grav. anomaly)

$R = 0$ (no anomaly)