PHYS 7810
Hydrodynamics
Spring 2024

Lecture 2
Stochastic differential equations

January 18

Recap: random walker moved $\Delta x = \pm \ell$ in $\Delta t = \tau$.

Position $\underset{= n\tau}{x(t)} = \sum_{k=1}^{n} \ell \cdot \underbrace{z_k}_{\pm 1} \rightarrow \sum_{k=1}^{n} \ell \sqrt{\tau} \cdot \underbrace{\xi((k-1)\tau)}_{\pm 1/\sqrt{\tau}}$

Probability $\mathbb{P}(x(t) = \tilde{x}) \sim \frac{1}{\sqrt{t}} \exp\left[- \frac{(\tilde{x}/\ell)^2}{2(t/\tau)} \right]$

$$\sim \frac{1}{\sqrt{t}} e^{-\tilde{x}^2/4Dt}$$

where diffusion const $D = \ell^2/2\tau$.

Goal: continuous time limit (i.e. $\ell, \tau \to 0$). Keep D finite.

$$\frac{x((n+1)\tau) - x(n\tau)}{\tau} \overset{?}{=} \frac{dx}{dt} = \frac{\ell}{\sqrt{\tau}} \xi(n\tau) = \sqrt{2D}\, \xi(n\tau)$$

Idea: $\dot{x}(t) = \frac{dx}{dt} = \sqrt{2D}\, \xi(t)$

But $\xi(t) = \{z_1, \dots, z_n\}$ is random.

In discrete setting: $\langle z_k \rangle = 0$, $\langle z_k z_{k'} \rangle = \delta_{kk'} = \begin{cases} 1 & k=k' \\ 0 & k \neq k' \end{cases}$.

$$\downarrow \qquad\qquad \downarrow$$

$$\langle \xi(t) \rangle = 0. \qquad \langle \xi(t) \xi(t') \rangle = \delta(t-t')$$

$$[\xi] = -\tfrac{1}{2}[t]$$

or: $\sum_{k'} \langle z_k z_{k'} \rangle = 1 \rightsquigarrow \int dt \langle \xi \xi \rangle = 1$

Put this together: stochastic differential equation

$$\frac{dx}{dt} = \sqrt{2D}\, \xi(t) \quad \text{where} \quad \begin{array}{l} \langle \xi(t) \rangle = 0 \\ \langle \xi(t) \xi(t') \rangle = \delta(t-t') \end{array}$$

Langevin equation

Gaussian white noise.

Example: using SDE to calculate correlation functions:

(1-point)

$$\langle x(t) \rangle = x(0)$$

$$\frac{d}{dt} \langle x(t) \rangle = \left\langle \frac{dx}{dt} \right\rangle = \langle \sqrt{2D}\, \xi(t) \rangle = 0.$$

$$\langle [x(t) - x(0)]^2 \rangle = \left\langle \left[\int_0^t ds\, \dot{x}(s) \right]^2 \right\rangle = \left\langle \left[\int_0^t ds \sqrt{2D}\, \xi(s) \right]^2 \right\rangle$$

$$= \int_0^t ds_1 \int_0^t ds_2\, 2D \underbrace{\langle \xi(s_1) \xi(s_2) \rangle}_{\delta(s_1 - s_2)} = \int_0^t ds_1\, 2D = 2Dt.$$

Above: we used $x(t) = x(0) + \int_0^t ds \sqrt{2D}\, \xi(s)$.

Can we generalize this idea?

e.g. $\dot{x} = \lambda x \xi(t) \rightsquigarrow x(t) \stackrel{?}{=} \exp\left[\lambda \int_0^t ds\, \xi(s) \right] x(0)$

Subtle: $x(dt) = x(0) + \lambda x \underbrace{\xi(t)\,dt}_{d\xi}$

Itō: $x(0)$

Stratonovich: $\cancel{x\left(\frac{dt}{2}\right)} = \frac{1}{2}[x(0) + x(dt)]$

Itō: $\langle x(dt)\rangle = x(0) + \cancel{\lambda x(0)\langle d\xi\rangle} = x(0)$.

Stratonovich: $x(dt) = x(0) + \frac{\lambda}{2}[x(0) + x(dt)]\,d\xi$

$$x(dt)\left[1 - \frac{\lambda}{2}d\xi\right] = x(0)\left[1 + \frac{\lambda}{2}d\xi\right]$$

$$\langle x(dt)\rangle = \left\langle x(0)\frac{1 + \frac{\lambda}{2}d\xi}{1 - \frac{\lambda}{2}d\xi}\right\rangle$$

$$= x(0)\langle 1 + \cancel{\lambda d\xi} + \lambda^2 d\xi^2 + \cdots\rangle$$

$\langle d\xi^2\rangle = dt.$

$$\langle x(dt)\rangle = x(0)(1 + \lambda^2 \cdot dt) \rightsquigarrow \frac{d\langle x\rangle}{dt} = \lambda^2\langle x\rangle.$$

Conclusion: SDE $\dot{x} = \lambda x \xi$ are not well-defined.
Need to specify prescription. $\rightsquigarrow$ Itō.

Bad: build effective theory... identify invariants for Lagrangian?
be well-defined!

Goal: "deterministic" solution to SDE. $(\to$ PDE$)$
(Crude: Schrödinger eq. is deterministic ...)

Calculate: probability density $P(x,t)$

$$\mathbb{P}[a \le x(t) \le b] = \int_a^b dx\, P(x,t)$$

Claim: Fokker-Planck equation

If $\dot{x} = a(x,t) + b(x,t)\xi(t)$, then

$$\partial_t P = -\partial_x(aP) + \frac{1}{2}\partial_x^2(b^2 P)$$ ← well-defined!

"Proof": Pick any function $f(x)$. Calculate

$$\frac{d}{dt}\langle f\rangle = \int dx\, f(x)\,\partial_t P$$

Pick $s<t$ — stochastic, known $\tilde{x}(s)$

On the other hand: $\frac{d}{dt}\langle f\rangle = \int dx\, P(x,s)\,\langle f(\tilde{x}(t)_s)\rangle$

Now $s \to t$ $t \to t+dt$: $\tilde{x}(t+dt) = \tilde{x}(t) + a\,dt + b\,d\xi$.

$$\langle f(t+dt)\rangle - \langle f(t)\rangle = \int dx\, P(x,t)\left\langle f(\tilde{x}(t) + a\,dt + b\,d\xi) - f(\tilde{x}(t))\right\rangle$$

$$= \int dx\, P\left\langle f'(\underbrace{\tilde{x}(t)}_{\text{known}})(a\,dt + b\,d\xi) + \frac{1}{2}f''(x)(a\,dt + b\,d\xi)^2 + \dots\right\rangle$$

Since $\underline{dt^2 \to 0}$. $\langle d\xi^2\rangle = dt$:

$$\frac{d}{dt}\langle f(t)\rangle = \int dx\, P\left(a\,\partial_x f + \frac{b^2}{2}\partial_x^2 f\right) + \cancel{\dots}^{\,0}$$ (for $\underline{\underline{\text{Gaussian}}}$)

$$\hookrightarrow = \int dx\, f\left[-\partial_x(aP) + \partial_x^2\left(\frac{b^2}{2}P\right)\right]$$

Thus $\partial_t P = -\partial_x(aP) + \partial_x^2\left(\frac{b^2}{2}P\right)$: FPE

Why neglect higher order? If ξ is Gaussian then

$$\langle d\xi^n\rangle = \int_{-\infty}^{\infty} d\tilde{\xi}\; \tilde{\xi}^n \frac{1}{\sqrt{2\pi\,dt}} e^{-\tilde{\xi}^2/2dt} \sim \begin{cases} 0 & n \text{ odd} \\ dt^n & n \text{ even.} \end{cases}$$

Non-Gaussian noise: $\langle d\xi^n\rangle \sim dt + \dots$

↳ higher-derivatives in FPE.

In this class: FPE $\gg$ Langevin (SDE)

FPE $\downarrow$ well-defined as written, no ambiguity

Solution to FPE has nice interpretation: if

SDE:

$P(x,0) = \delta(x-y)$ [deterministic start at $x(0)=y$]

$P(x,t)$ = prob. dens. of $x(t)$ as random var.

Analogy to QM: $\langle x|\psi(t)\rangle \sim \langle x|e^{-iHt}|y\rangle$

Define: $|P(t)\rangle = \int dx\, P(x,t)\,|x\rangle$ $\qquad [\langle x|y\rangle = \delta(x-y)]$.

"Hamiltonian"

Abstract FPE: $\partial_t |P\rangle = -\hat{W}|P\rangle$

if $\hat{W}$ is t-ind. $\rightsquigarrow$ $|P(t)\rangle = e^{-\hat{W}t}|P(0)\rangle$

$P(x,t)$ above: $\langle x|e^{-\hat{W}t}|y\rangle$

Be careful: $\hat{W}$ is not Hermitian.

Generalize this derivation to n DoF: $i=1,\ldots,n$

$$\dot{x}_i = a_i(x) + b_{i\alpha}\xi_\alpha(t)$$

Gaussian white noise

$\langle \xi_\alpha(t)\,\xi_\beta(t')\rangle = \delta_{\alpha\beta}\,\delta(t-t')$

$\langle \xi_\alpha\rangle = 0$.

Itô FPE:

$$\partial_t P(x_i,t) = -\hat{W}P$$
$$= -\partial_i(a_i P) + \tfrac{1}{2}\partial_i\partial_j(b_{i\alpha}b_{j\alpha}P)$$

prob. dens. in n dim.

Einstein sum on repeated indices

Example: random walk, in continuous time:

$$\dot{x} = \sqrt{2D}\,\xi(t)$$

FPE $\downarrow$

$$\partial_t P = \frac{1}{2}\partial_x^2(2D \cdot P) = D\,\partial_x^2 P \qquad \text{(diffusion eq.)}$$

Solve by Fourier transform in x:

$$\hat{P}(k,t) = \int_{-\infty}^{\infty} dx\, e^{-ikx} P(x,t) \quad : \qquad \partial_x \to ik$$

$$\partial_t \hat{P} = -Dk^2 \hat{P} \qquad \text{so} \qquad \hat{P}(k,t) = \hat{P}(k,0)\,e^{-Dk^2 t}.$$

If we start at $x = y$:

$$\hat{P}(k,0) = \int dx\, e^{-ikx}\,\delta(x-y) = e^{-iky}$$

$$P(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-Dk^2 t}\, e^{-iky} \cdot e^{ikx}$$

$$= \frac{1}{\sqrt{4\pi D t}}\, e^{-(x-y)^2/4Dt}$$

$\leftarrow$ diffusive, Gaussian spread from random walk.