PHYS 7810
Hydrodynamics
Spring 2024

Lecture 20
Electron hydrodynamics

April 2

Today: consider a system where one (or more) conservation laws are weakly broken.

e.g. mass & momentum conservation... (assume Galilean...)

mass ↓

$$\partial_t \rho + \partial_i(\rho v_i) = 0$$

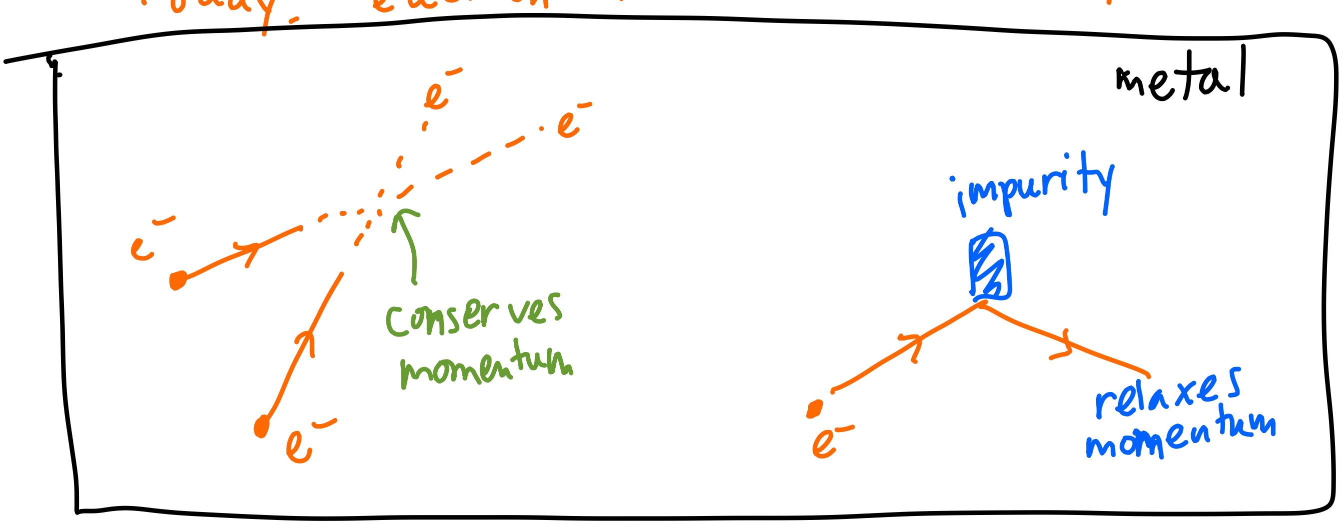
momentum ↓

$$\partial_t(\rho v_i) + \partial_j(P\delta_{ij} + \rho v_i v_j - \text{viscous}) = 0 \quad \to \quad = -\gamma \rho v_i$$

γ = momentum relaxation rate

Physical origins? Atmospheric flows, friction btwn air & ground

Today: electron motion in solid-state physics



In general, no momentum conservation... no viscous hydrodynamics.

What are consequences of $\gamma \neq 0$?

(1) Quasinormal modes $\longmapsto$ "Quasihydrodynamic"

not all of the DOF $\{v_i\}$ have arbitrarily long relaxation times

let's assume in equilibrium: $\rho \approx \bar{\rho}$ (const.) $\bar{v}_i = 0$.

Then if $\rho = \bar{\rho} + \delta\rho$, $v_i = \bar{v}_i + \delta v_i$:

$$\partial_t \delta\rho + \bar{\rho}\, \partial_i \delta v_i = 0$$

$$\bar{\rho}\, \partial_t \delta v_i + \left.\frac{\partial P}{\partial \rho}\right|_{\bar{\rho}} \partial_i \delta\rho - \eta\, \partial_j \left[\partial_i \delta v_j + \partial_j \delta v_i - \frac{2}{d}\delta_{ij}\partial_k \delta v_k\right] - \zeta\, \partial_j \partial_i \delta v_j = -\gamma \bar{\rho}\, \delta v_i$$

Look for plane wave solutions $\propto e^{ikx - i\omega t}$

$$\begin{cases} i\omega\, \delta\rho = ik\bar{\rho}\, \delta v_x \\ i\omega\, \delta v_x = \frac{1}{\bar{\rho}}\frac{\partial P}{\partial \rho} ik\, \delta\rho + \left[\underbrace{\frac{1}{\bar{\rho}}\left(\zeta + \frac{2d-2}{d}\eta\right)}_{\nu_{\text{eff}}} k^2 + \gamma\right]\delta v_x \end{cases}$$

$$\underbrace{i\omega\, \delta v_\perp = \left[\frac{1}{\bar{\rho}}\eta k^2 + \gamma\right]\delta v_\perp}$$

$\hookrightarrow$ $\omega = -i\left[\gamma + \frac{\eta}{\bar{\rho}}k^2\right] = -i\left[\gamma + \nu k^2\right]$

"diffusion" of transverse momentum (lec 8, 9)...
plus constant decay γ.

$$\frac{\delta\rho}{\bar{\rho}} = \frac{k}{\omega}\delta v_x \qquad \text{and} \qquad \frac{\partial P}{\partial \rho} = v_s^2$$

$$\omega = \frac{k^2 v_s^2}{\omega} - i\left[\gamma + \nu_{\text{eff}} k^2\right]$$

distorted sound waves

$$\omega = \frac{-i\left[\gamma + \nu_{\text{eff}} k^2\right] \pm \sqrt{(2kv_s)^2 - \left[\gamma + \nu_{\text{eff}} k^2\right]^2}}{2}$$

As $k \to 0$:

$$\omega \approx -\frac{i}{2}\left(\gamma + \nu_{\text{eff}} k^2\right)\left[1 \pm \sqrt{1 - \left(\frac{2kv_s}{\gamma}\right)^2}\right]$$

$$\approx \begin{cases} -i\gamma & (+) \\ -ik^2 \cdot D & (-) \end{cases}$$

where $D = v_s^2/\gamma$ is an emergent diffusion constant!

Sound mode $\longrightarrow$ diffusion on longest scales $(k \to 0)$.

True hydrodynamics $(k \to 0)$: $\omega = -iDk^2$.

Which mode survives? If $\omega \ll \gamma$

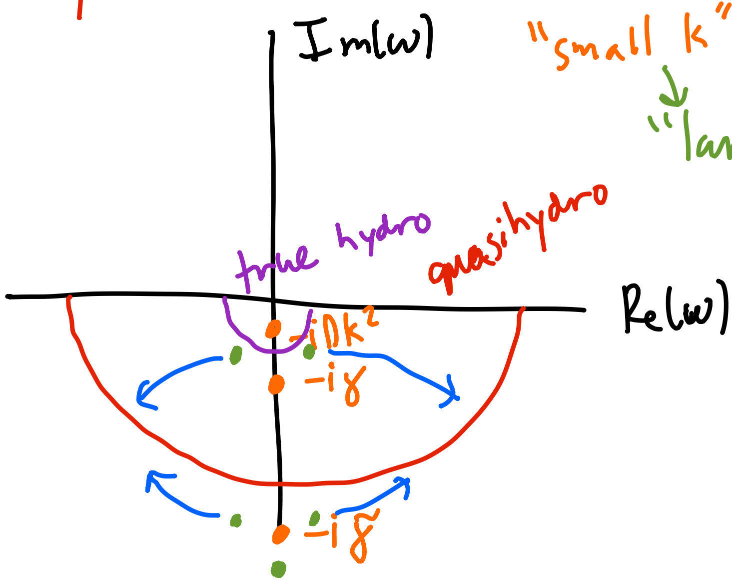
$$\cancel{\partial_t \delta v_i}_{\omega \ll \gamma} + \partial_i \frac{\partial P}{\partial \rho} \frac{\delta \rho}{\bar{\rho}} + \cdots = -\gamma\, \delta v_i, \quad \text{so} \quad \frac{\delta \rho}{\bar{\rho}} \approx -\frac{\gamma}{ik} \delta v_x$$

$$\partial_t \delta\rho + ik\bar{\rho}\, \delta v_x = \partial_t \delta\rho + \overbrace{\frac{v_s^2}{\gamma}}^{D} k^2 \delta\rho = 0$$

The true hydro mode = charge diffusion!

Quasihydro = both charge & slow momentum.

Dynamics both when $\omega \ll \gamma$ AND $\tilde{\gamma} \gg \omega \gg \gamma$ ← other microscopic scales...



"small k"

"larger k"

"larger still k" when $\omega \sim \tilde{\gamma}$ (quasihydro breaks...)

Hierarchy of scattering rates $\gamma \ll \tilde{\gamma}$.

Such a hierarchy can exist! e.g. in graphene (2d lattice of carbon)

momentum relaxing (el-imp) $\quad \gamma \sim 3\times 10^{11}\, s^{-1} \rightsquigarrow \ell = \frac{v_F}{\gamma} \sim 3\times 10^{-6}\, m$

momentum conserving (non-hydro) $\quad \tilde{\gamma} \sim 10^{13}\, s^{-1} \rightsquigarrow \tilde{\ell} \sim 10^{-7}\, m$

Intuitively, see viscous flow if $\tilde{\gamma} \gg \omega \gg \gamma$.
(intermediate time scales!)

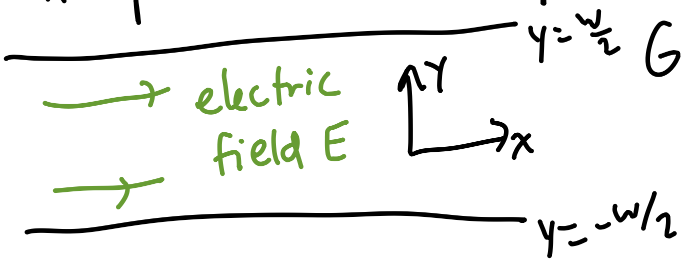
For electron liquid, experimental probes are electronic:
what's voltage drop? what's electrical resistance?

Because electrons are charged, natural to work w/

$$\rho \to \frac{\text{charge density}}{-e} \qquad \frac{\partial P}{\partial \rho} = -e \frac{\partial P}{\partial(-e\rho)} = -e\mu_{ec}$$

μ_{ec} ← electrochemical potential ← voltage

Example: Poiseuille / pipe flow (cf. lec 13)



Guess: no-slip boundary condition
$v_x = 0$ at $y = \pm w/2$.

$$\partial_x P - \eta \partial_y^2 v_x = -\gamma \rho v_x \cdot m$$

$\hookrightarrow \partial_x P = \rho \partial_x \mu_{ec} \to -e\rho(-E)$
(using thermo cf lec 8)

(effective) mass of electron

Hence: $-e\rho E = m\rho\gamma v_x - \eta \partial_y^2 v_x$

Solve by defining $\underbrace{\lambda = \sqrt{\eta / \gamma m}}_{\text{Gurzhi length}}$ and $v_x = \tilde{v}_x(y) - \frac{eE}{\gamma m}$

$$\downarrow$$

$$0 = \tilde{v}_x - \lambda^2 \partial_y^2 \tilde{v}_x$$

Solved by $\tilde{v}_x = Ae^{-y/\lambda} + Be^{+y/\lambda}$. Expect $A = B$.

So $v_x = -\frac{eE}{\gamma m} + 2A\cosh\frac{y}{\lambda}$. Impose no-slip boundary condition

$$v_x(y) = -\frac{eE}{\gamma m}\left[1 - \frac{\cosh(y/\lambda)}{\cosh\left(\frac{W}{2\lambda}\right)}\right].$$

Flow qualitatively change depending on W/λ.

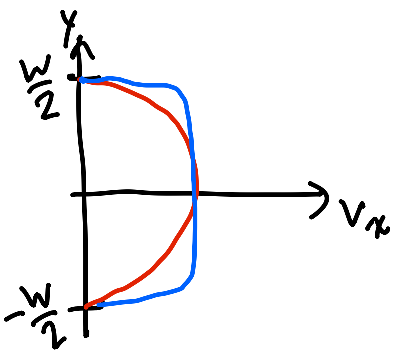
(quasi)hydro (viscous): $w \ll \lambda$ so

$$v_x(y) \approx -\frac{eE}{\gamma m}\left[1 - \frac{1+y^2/2\lambda^2+\cdots}{1+w^2/8\lambda^2+\cdots}\right] \approx -\frac{eE}{m\gamma}\,\frac{1}{2\lambda^2}\left[\left(\frac{w}{2}\right)^2 - y^2\right]$$

This is the same as usual Poiseuille flow. $\hookrightarrow = -\frac{eE}{2\eta}\left[\cdots\right]$.

Diffusive/Ohmic/"true hydro": $w \gg \lambda$:

up to region $\left|\frac{w}{2}\right| - |y| \lesssim \lambda$ ... $v_x(y) \approx -\frac{eE}{\gamma m}$.



Flow profile inhomogeneous in quasihydro regime ... homogeneous in Ohmic.

What is most directly measured is electrical resistance.

$$R = \frac{V}{I} = \frac{E\cdot \overset{\text{length}}{L}}{-e\int_{-w/2}^{w/2} dy\, \rho v_x} \approx \begin{cases} \frac{12\eta}{\rho e^2}\frac{L}{w^3} & w \ll \lambda \\ \frac{\gamma m}{\rho e^2}\frac{L}{w} & w \gg \lambda \ \text{Ohmic} \end{cases}$$

$\frac{\gamma m}{\rho e^2} \to \frac{1}{\sigma_{dc}} \to$ dc electrical conductivity $\to$ Drude formula

textbook EM formula:

$$R = \rho_{el}\frac{L}{\text{cross-sec. area}}$$

In quasihydro/viscous limit: $\frac{12\eta}{\rho e^2}\frac{1}{w^2} \sim \frac{1}{\sigma_{eff}}$ → width-dependent.

This w-dependence is expt'l signature ... seen in multiple materials (GaAs, WP_2 ...)

Problem: hard to deduce boundary conditions.

e.g. if no-stress ($\tau_{xy} = 0$) $\rightsquigarrow$ $\partial_y v_x = 0$

This case: consistent solution $v_x = -\frac{eE}{\gamma m}$. η-indep. / λ-indep.

Main challenge in finding viscous quasihydro of electrons:

$\frac{\gamma}{\tilde{\gamma}} \gtrsim 10^{-2}$ ... is this good enough separation of scales?

↳ cf lecs 21–24: $\eta/_{mp} \sim \frac{v_F^2}{\tilde{\gamma}}$, so $\lambda \sim \frac{v_F}{\sqrt{\gamma\tilde{\gamma}}} \sim \sqrt{\ell\tilde{\ell}}$

geometric mean of mean free paths

quasihydro: look at $\tilde{\ell} \ll w \ll \lambda = 10\tilde{\ell}$ ...

Possible application: anisotropic hydro?

↳ anisotropic Fermi surfaces.

single-particle kinetic energy $\varepsilon(\vec{p}) \neq \frac{\vec{p}^2}{2m}$

$$= \frac{\vec{p}^2}{2m} + \underbrace{\alpha[p_x^4 + p_y^4]}_{\text{break rotational sym... (anisotropic viscosity...)}} + \dots$$

AND breaks Galilean invariance!

$$J_i = \rho v_i \underbrace{- \sigma_0 \partial_i \mu_{th}}_{\text{"incoherent conductivity"}} + \dots$$

Open problem: detect these in expt!