PHYS 7810
Hydrodynamics
Spring 2024

Lecture 21
The BBGKY hierarchy

April 4

So far we've discussed hydrodynamics as an effective field theory

Now, kinetic theory: a "microscopic derivation" of hydro for gases / systems with weak interactions

Why? Alternate/historical perspective $\rightsquigarrow$ "first principles" calculation of viscosity! a window into beyond-hydro regime...

Starting point: weakly interacting classical Hamiltonian system:

external potential

$$H = \sum_{\alpha=1}^{N} \left[\frac{\vec{p}_\alpha^{\,2}}{2m} + U(\vec{x}_\alpha) \right] + \sum_{\alpha<\beta} V_2(|\vec{x}_\alpha - \vec{x}_\beta|)$$

two-body interaction

Can replace w/ anisotropic (Lec 20, HW6)

which particle $\alpha = 1, \dots, N$

Use Poisson bracket: $\underbrace{\{x_{i\alpha}, p_{j\beta}\}}_{\text{collect into phase space } \xi^I_\alpha = (x_{i\alpha}, p_{i\alpha})} = \delta_{\alpha\beta}\delta_{ij}$ which spatial dimension $i = 1 \dots d$.

Goal: Solve for phase space probability distribution $P(\xi, t)$.

Follow lec 3-5 ... Fokker-Planck equation:

$$\partial_t P = -\sum_{\alpha I} \frac{\partial}{\partial \xi^I_\alpha}\left(\{\xi^I_\alpha, H\} P\right) \qquad \text{(aka Liouville's equation)}$$

Is exact! But unwieldy... too high-dimensional, not dissipative...

Idea: Kinetic theory $\rightsquigarrow$ clever "truncation" of FPE.

In particular... when interactions are weak, most of experimental observables captured by one-particle distribution function

$$f_1(\xi^I, t) = \left\langle \sum_{\alpha=1}^{N} \delta(\xi^I - \xi^I_\alpha) \right\rangle = \int (d^{2dN}\xi)\, P(\xi, t) \sum_\alpha \delta(\xi^I - \xi^I_\alpha)$$

often just called $\longrightarrow f$.

For example:

particle number density $\quad \rho(x,t) = \int d^d p\, f_1(x,p,t)$

momentum density $\quad g_i(x,t) = \int d^d p\, f_1 \cdot p_i$

Goal: find an approximate (?) equation just for f_1.

Useful to assume initial condition on

$$P(\xi^I_1, \xi^I_2, \ldots) = P(\xi^I_2, \xi^I_1, \ldots) \quad \text{(permutation-symmetric)}$$

$\to$ "indistinguishable particles!"

Since H is perm. symmetric...

P is perm. symmetric for all t.

[Assuming that one species of particles... "easy" to generalize]

So capture P via... $\quad f_1(\xi_1) = N \int d^{2d}\xi_2 \cdots d^{2d}\xi_N\, P$

For later... we'll want n-particle correlations:

$$f_n(\xi_1, \ldots, \xi_n) = \frac{N!}{(N-n)!} \int d^{2d}\xi_{n+1} \cdots d^{2d}\xi_N\, P$$

Now, calculate exactly f_1's equation of motion:

$$\partial_t f_1 = N\int d^{2d}\xi_2 \cdots \partial_t P = N\int d^{2d}\xi_2\cdots\left[-\sum_{\alpha=1}^{N}\frac{\partial}{\partial \xi_\alpha^I}\left(\{\xi_\alpha^I, H\}P\right)\right]$$

All of terms are total derivatives... except for $\partial/\partial\xi_1^I$:

$$\partial_t f_1 = -N\frac{\partial}{\partial \xi_1^I}\int d^{2d}\xi_2\cdots \underbrace{\{\xi_1^I, H\}}\, P$$

Use Hamilton's equation:

$$\{x_{i1}, H\} = \frac{\partial H}{\partial p_{i1}} = \frac{p_{i1}}{m}$$

$$\{p_{i1}, H\} = -\frac{\partial H}{\partial x_{i1}} = -\frac{\partial U(x_1)}{\partial x_{i1}} - \sum_{\alpha=2}^{N}\frac{\partial}{\partial x_{i1}} V_2(|\vec{x}_1 - \vec{x}_\alpha|)$$

$$\partial_t f_1 = -N\int d^{2d}\xi_2\cdots\Bigg[\underbrace{\frac{\partial}{\partial x_{i1}}\left(\frac{p_{i1}}{m}P\right) + \frac{\partial}{\partial p_{i1}}\left(-\frac{\partial U(x_1)}{\partial x_{i1}}P\right.}\ \underbrace{\left. - \sum_{\alpha=2}^{N}\frac{\partial V_2(x_1 - x_\alpha)}{\partial x_{i1}}P\right)}\Bigg]$$

integrate out $2\ldots N$:

$$-\frac{\partial}{\partial x_{i1}}\left(\frac{p_{i1}}{m}f_1\right) + \frac{\partial}{\partial p_{i1}}\left(\frac{\partial U}{\partial x_{i1}}f_1\right) \qquad + \frac{\partial V_2(x_1 - x_2)}{\partial x_{i1}} f_2(\xi_1, \xi_2)$$

$\Rightarrow$ streaming terms $\qquad\qquad$ $\Rightarrow$ collision term/integral

$$\partial_t f_1 + \frac{p_{i1}}{m}\frac{\partial f_1}{\partial x_{i1}} - \frac{\partial U}{\partial x_{i1}}\frac{\partial f_1}{\partial p_{i1}} = \int d^{2d}\xi_2\, \frac{\partial V_2(\vec{x}_1 - \vec{x}_2)}{\partial x_{i1}}\frac{\partial f_2}{\partial p_{i1}}$$

Hence, exact LINEAR equation for f_1

but it depends on f_2.

Repeat the above argument... :

$$\partial_t f_n + \{f_n, H_n\} = \sum_{\alpha=1}^{n}\int d^{2d}\xi_{n+1}\frac{\partial V_2(x_\alpha - x_{n+1})}{\partial x_{i\alpha}}\frac{\partial f_{n+1}}{\partial p_{i\alpha}}$$

$\leftarrow$ Ham. of first n particles

These equations are called the BBGKY Hierarchy.

Goal: how to truncate equations into something manageable?

Focus on $n=1,2$: (drop spatial indices ... $(d=1)$)

$$\partial_t f_1 + \underbrace{\frac{p_1}{m}\frac{\partial f}{\partial x_1} - \frac{\partial U}{\partial x_1}\frac{\partial f}{\partial p_1}} = \underbrace{\int d\xi_2 \, \frac{\partial V_2(x_1-x_2)}{\partial x_1}\frac{\partial f}{\partial p_1}}$$

$$\partial_t f_2 + \underbrace{\frac{p_1}{m}\frac{\partial f_2}{\partial x_1} - \frac{\partial U}{\partial x_1}\frac{\partial f_2}{\partial p_1} + \frac{p_2}{m}\frac{\partial f_2}{\partial x_2} - \frac{\partial U}{\partial x_2}\frac{\partial f}{\partial p_2}} - \underbrace{\frac{\partial V_2(x_1-x_2)}{\partial x_1}\left(\frac{\partial f}{\partial p_1} - \frac{\partial f}{\partial p_2}\right)}$$

$$= \underbrace{\int d\xi_3 \left[\frac{\partial V_2(x_1-x_3)}{\partial x_1}\frac{\partial f_3}{\partial p_1} + \frac{\partial V_2(x_2-x_3)}{\partial x_2}\frac{\partial f_3}{\partial p_2}\right]}$$

There's a number of time scales in this problem...

single particle: $\sim f/\tau_1$ collision: $\sim$ "f"$/\tau_c$ two-particle $\sim f_2/\tau_2$.

For simplicity... take $U=0$. $\bar{p} \sim$ typical momentum

$L^{-1} \sim \dfrac{\partial_x f_1}{f_1}$ (more later...)

$$\frac{1}{\tau_1} \sim \frac{\bar{p}}{m}\cdot\frac{1}{L}$$

interaction is short range on length scale a

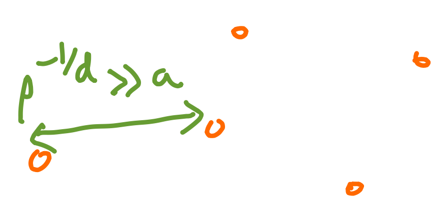
$$1/\tau_2 \sim \frac{1}{\bar{p}}\frac{V_2}{a}$$

$$1/\tau_c \sim \frac{V_2}{a}\frac{1}{\bar{p}}\left[\int_{|x_2-x_1|\lesssim a} d\xi_2 \, f_2\right] \sim \frac{1}{\tau_2}\cdot \rho a^d$$

ρ ← density of particles

Now, $1/\tau_1$ can be big $(L\to\infty)$

In a gas, we expect $\rho a^d \ll 1$:



So $\underline{1/\tau_c \ll 1/\tau_2}$

In a reasonable "steady-state-like" approximation:

$$0 \approx -\partial V_2(x_1-x_2)/\partial x_1\left(\frac{\partial f_2}{\partial p_1} - \frac{\partial f_2}{\partial p_2}\right) + \frac{p_1}{m}\frac{\partial f_2}{\partial x_1} + \frac{p_2}{m}\frac{\partial f_2}{\partial x_2}.$$

Switch to coordinates: $\tilde{x} = x_1 - x_2$ $\quad \bar{x} = \dfrac{x_1 + x_2}{2}$

$\tilde{p} = p_1 - p_2$ $\quad \bar{p} = \dfrac{p_1 + p_2}{2}$

$$\underline{\frac{\partial}{\partial \bar{x}} \sim 1/L \quad \text{but} \quad \frac{\partial}{\partial \tilde{x}} \sim 1/a .}$$

Hence: $$0 \approx -\frac{\partial V_2}{\partial \tilde{x}} \frac{\partial f_2}{\partial \tilde{p}} + \frac{\tilde{p}}{m} \frac{\partial f_2}{\partial \tilde{x}} + \cancel{\frac{\bar{p}}{m} \frac{\partial f_2}{\partial \bar{x}}}$$

Now we have:
$$\partial_t f_1 + \frac{p_1}{m} \frac{\partial f_1}{\partial x_1} \approx \int dp_2 \, dx_2 \left[-\frac{\tilde{p}}{m} \frac{\partial f_2}{\partial \tilde{x}} \right]$$

Naive integral over x_2 gives 0?? $\approx \dfrac{\partial f_2}{\partial x_2}$?

$$\int dp_2 \, dx_2 \, \frac{\tilde{p}}{m} \frac{\partial f_2}{\partial x_2} = \int dp_2 \, \frac{\tilde{p}}{m} \Big[f_2(2 \text{ right of } 1) - f_2(2 \text{ left of } 1) \Big]$$

On physical grounds... suppose $\tilde{p} > 0$ $\quad (p_1 > p_2)$

collision in future!

collision in past!

Let's focus on future collision terms...

Make assumption/ansatz/postulate of <u>molecular chaos</u>:

<u>before</u> collision: $\quad f_2^{\text{before}}(\xi_1, \xi_2) \approx f_1(\xi_1) f_1(\xi_2)$

i.e. particles uncorrelated <u>before</u> collision!
become correlated by collision.

$$\int dp_2 \, \frac{\tilde{p}}{m} f_2[2 \text{ right of } 1] \to \text{“}\int d\tilde{p} \, \frac{\tilde{p}}{m}\text{''} \, f_1(\overset{x_1, p_1}{\xi_1}) f_1(\overset{x_1, p_2}{\underline{\cancel{\xi_2}}}) \; ?$$

but no integral over x_2...

Idea: $\xi_2 \approx (x_1, p_2)$ b/c interaction is short range.

Now: $\partial_t f_1 + \frac{p_1}{m}\frac{\partial f_1}{\partial x_1} = \underbrace{\mathcal{C}[f_1]}_{\text{collision integral}}$

$$= -\int dp_2 \left[\text{Rate}(p_1 p_2 \to \text{any?})\, f_1(x_1,p_1)\, f_1(x_1,p_2) - \text{collisions in past}\right]$$

$$= -\int dp_2\, dp_1'\, dp_2' \Big[\text{Rate}(p_1 p_2 \to p_1' p_2')\, f_1(p_1) f_1(p_2) - \text{Rate}(p_1' p_2' \to p_1 p_2)\, f_1(p_1') f_1(p_2')\Big]$$

loss of p_1's

gain of p_1's.

e.g. scattering matrix/rate from QFT...

This equation is called Boltzmann equation / Kinetic equation.

Generalizing to d spatial dimensions easy...

In quantum systems: the collision integral:

$$= -\int dp_2\, dp_1'\, dp_2' \Big[\text{Rate}(p_1 p_2 \to p_1' p_2')\, f_1(p_1) f_1(p_2) \times \begin{cases} (1-f_1(p_1'))(1-f_1(p_2')) \\ (1+f_1(p_1'))(1+f_1(p_2')) \end{cases}$$

Pauli blocking ← fermions

Stimulated emission! ← bosons

If add external potential:

$$\partial_t f + \frac{p_i}{m}\frac{\partial f}{\partial x_i} - \frac{\partial U}{\partial x_i}\frac{\partial f}{\partial p_i} = \mathcal{C}[f]$$

same as before!

$f_1 \to f$