PHYS 7810
Hydrodynamics
Spring 2024

Lecture 22
The Boltzmann equation

April 9

Recall: start w/ hydro as EFT ...
today: kinetic theory.

So far: Reduce N-particle distribution f_N
↳ $f_1(x,p) = f(x,p) = N \times$ probability density in phase space

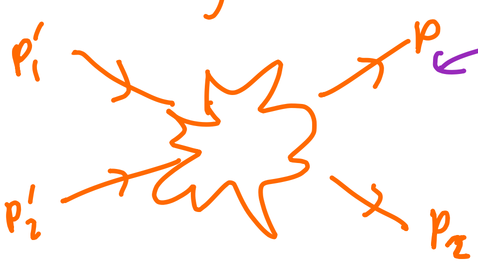
Using molecular chaos approximation → Boltzmann equation:

single-particle Ham. ; collision integral

$$\partial_t f + \partial_{p_i} H_1 \partial_{x_i} f - \partial_{x_i} H_1 \partial_{p_i} f = C[f]$$

collision rate ; particle creation ; particle loss ; all @ same point

$$C[f] = \int dp_2\, dp_1'\, dp_2'\, \mathcal{R}(p_1' p_2' \to p p_2)\left[f(p_1',x) f(p_2',x) - f(p,x) f(p_2,x)\right]$$



- assumptions of 2-body interactions
- if boson vs. fermions
  - boson ↓ $(1+f)$
  - fermions ↓ $(1-f)$

Goal:
1. irreversibility / arrow of time?
2. equilibrium?
3. ideal hydro [how to solve]

① Irreversible b/c of "molecular chaos"
Quantified via "H Theorem"

Claim: $-H = S = \int d^dx\, d^dp\, [-f \log f]$ obeys $\frac{dS}{dt} \geq 0$.

This is 2nd law of thermo!

$$-\frac{d}{dt}\underbrace{\int d^dx\, d^dp\, f}_{N} = 0$$

Proof: $$\frac{dS}{dt} = \int d^dx\, d^dp\, \frac{\partial f}{\partial t}\left[-\log f - \cancel{f \cdot \frac{1}{f}}\right]$$

(Poisson bracket)

$$= -\int d^dx\, d^dp \Big(C[f] - \underbrace{\{f, H_1\}}_{\text{if } H_1 = \varepsilon(p) + \cancel{U(x)}}\Big)\log f$$

$$\int d^dx\, d^dp\, \{f, H_1\}\log f = \int d^dx\, d^dp\, \frac{\partial f}{\partial x_i}\frac{\partial \varepsilon}{\partial p_i}\log f$$

$$= -\int d^dx\, d^dp\, f\,\frac{\partial}{\partial x_i}\left[\frac{\partial \varepsilon}{\partial p_i}\log f\right]$$

$$= -\int d^dx\, d^dp\, \cancel{f}\,\frac{\partial \varepsilon}{\partial p_i}\frac{1}{\cancel{f}}\frac{\partial f}{\partial x_i} = \int d^dx\, d^dp\, f\frac{\partial^2 \varepsilon}{\partial x \partial p} = 0.$$

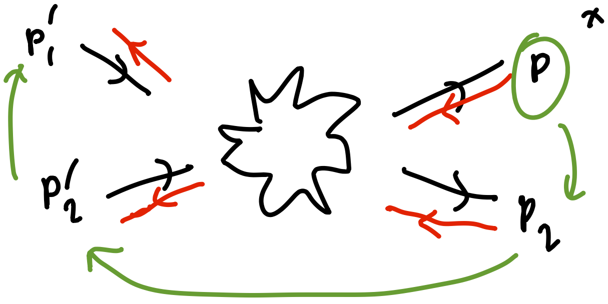
$$= -\int d^dx\, d^dp\, C[f]\log f$$

entropy produced by collisions

$$= -\int d^dx\, d^dp\, d^dp_2\, d^dp_1'\, d^dp_2'\, R(p_1'p_2' \to pp_2)\left[f(p_1')f(p_2') - f(p)f(p_2)\right]\log f(p).$$

Key idea: assume has [time-reversal + inversion] symmetry:

$p \to -p$, $x \to x$ $\qquad$ $p \to -p$, $x \to -x$



$$R(p_1'p_2' \to p\, p_2) = R(pp_2 \to p_1'p_2')$$

$$= -\frac{1}{4}\int d^dx\, d^dp \cdots d^dp_2'\, R(p_1'p_2' \to pp_2)\left[f(p_1')f(p_2') - f(p)f(p_2)\right]$$

$$\times \underbrace{\left[\log f(p) + \log f(p_2) - \log f(p_1') - \log f(p_2')\right]}_{\left[\log[\overset{X}{f(p)f(p_2)}] - \log[\overset{Y}{f(p_1')f(p_2')}]\right]}$$

$= +\frac{1}{4}\int d^dx \cdots R\underbrace{[X-Y][\log X - \log Y]}_{\geq 0 \text{ b/c } \log \text{ is monotonically increasing.}}$

Thus $\frac{dS}{dt} \geq 0$. $\Rightarrow$ 2^{nd} Law of thermo! (H-Thm)

(2) What are equilibria of Boltzmann eq.?

$\hookrightarrow \frac{\partial f}{\partial t} = 0.$

H-Thm: $\frac{dS}{dt} = 0 = \frac{1}{2}\int d^dp\, d^dp_2\, d^dp_1'\, d^dp_2'\, R(p_1'p_2' \to pp_2)[f(p_1')\cdots]$
$\times [\log[f(p_1')\cdots] - \log[f(p)f(p_2)]]$

$R \geq 0$: either $R = 0$ or $\underline{f(p_1')f(p_2') = f(p)f(p_2)}$

$\underbrace{\log}_{Z} f(p_1') + \log f(p_2') = \log f(p) + \log f(p_2).$

$Z(p_1') + Z(p_2') = Z(p) + Z(p_2)$ $\underline{\text{if } R \neq 0.}$

$\hookrightarrow$ suggests: Z is conserved: $Z = 1$ particle
$Z = p_i$ momentum
$Z = \varepsilon$ energy

Physical grounds: $R(p_1'p_2' \to pp_2) = \delta(p_1' + p_2' - p - p_2)\,\delta(\varepsilon(p_1') + \varepsilon(p_2') - \varepsilon(p) - \varepsilon(p_2)) \times \cdots$

Conclusion: equilibria include

$$f(p) = \exp[-\underline{\beta}(\varepsilon(p) - \underline{\mu_{th}} - \underset{\text{const.}}{\underline{\vec{v}}} \cdot \vec{p})]$$

generalized Gibbs / grand canonical ensemble!

What about $\beta(x)$? NO! streaming terms $\neq 0$.
$(\{f, H_1\})$

(3) Time-dependent solutions? Hard in general!

A first question: on long length scales $\rightarrow$ hydrodynamic?

Yes! (thru Lec 23)

Today: recovery of ideal hydro (dissipation-free)

Check: Boltzmann equation conserve mass/momentum/energy.

$\hookrightarrow Z = \{1, p_i, \varepsilon\}$

$$\frac{d}{dt}\int d^dx\, d^dp\, Z(p) f(x,p,t) = 0.$$

x_i – total derivative

$$= \int d^dx\, d^dp\, Z(p)\left[-\cancel{\frac{\partial \varepsilon}{\partial p_i}\frac{\partial f}{\partial x_i}}^{\,0} + \mathcal{C}[f]\right]$$

E.g. $Z = p_i$:

$$\int d^dp\, d^dp_2\, d^dp_1'\, d^dp_2'\; R(p_1'p_2' \rightarrow p p_2)\left[f(p_1')f(p_2') - f(p)f(p_2)\right] p_i$$

Use relabeling trick: $\int \cdots p_i = \frac{1}{4}\int \cdots \underbrace{[p_i + p_{2i} - p_{1i}' - p_{2i}']}_{R \sim \delta(p + p_2 - p_1' - p_2')} = 0$

(i.e. momentum conserved)

Hence, hydro DOF make sense. Conserved mass/mom./energy.

Recall our equilibria: $f = \exp[-\beta(\varepsilon - \mu_n - v_i p_i)]$

Expect: "hydro solution" $\approx$ $f(x,p) \approx \exp[-\beta(x)(\varepsilon - \mu_n(x) - v_i(x) p_i)]$

Goal: plug this guess into Boltzmann...

$\hookrightarrow \partial_t n + \partial_i J_i \overset{?}{=} 0$

Calculate $n(x,t) = \int d^dp\; 1 \cdot f(x,p,t)$

$$g_i(x,t) = \int d^dp\; p_i \cdot f(x,p,t)$$

$$\varepsilon(x,t) = \int d^dp\; \frac{p^2}{2m} f(x,p,t) \qquad \text{[if Gal. inv.]}$$

Show that: $\int d^d p \; e^{-\beta(p^2/2m - \mu_{th} - v_i p_i)} = \left(\frac{2\pi m}{\beta}\right)^{d/2} e^{\underbrace{\beta(\mu_{th} + \frac{m}{2}v^2)}_{\text{Lec 9: Gal inv. req.}}} = n$

$g_i = \int d^d p \; p_i \cdots = m n v_i$

Recovering thermodynamics!

$$\partial_t n = \int d^d p \, \partial_t f = \int d^d p \left[\cancel{C[f]} - \frac{\partial \varepsilon}{\partial p_i}\frac{\partial f}{\partial x_i} \right] = -\frac{\partial}{\partial x_i}\int d^d p \; V_i f$$

$\cancel{C[f]} \to 0$

$V_i = \frac{\partial \varepsilon}{\partial p_i} = \frac{p_i}{m}$

$\hookrightarrow \partial_t n + \frac{\partial}{\partial x_i} J_i = 0$ $\quad J_i = \int d^d p \; V_i f =$ ~~number~~ particle current

by Gal. $J_i = \frac{1}{m} g_i$

Similar arguments:

$$\partial_t g_i + \frac{\partial}{\partial x_j}\underbrace{\int d^d p \; V_j p_i f}_{} = 0$$

stress tensor $T_{ji} = \int d^d p \; \frac{p_j p_i}{m} e^{\cdots} = n\left[\frac{1}{\beta}\delta_{ij} + m v_i v_j\right]$

$P = \frac{n}{\beta} = nT$ $\quad$ ideal gas law

thermo sanity check: $\quad n = \frac{\partial P}{\partial \mu_{th}}$ ✓

Repeat for energy:

energy current $\mathcal{E}_i = (\varepsilon + P) v_i$ $\quad$ where $\varepsilon = \frac{d}{2}P + \frac{mn}{2}v^2$

Recap: by neglecting $C[f]$ $\quad$ (choosing $f = e^{-\beta\varepsilon\cdots}$ ansatz)

$\hookrightarrow \int d^d p \; Z \, \partial_t f \longrightarrow$ hydro eq. w/o dissipation

Lec 23: studying corrections $\rightsquigarrow$ viscous hydro

24: beyond hydro