PHYS 7810
Hydrodynamics
Spring 2024

Lecture 23
Hydrodynamics from kinetic theory

April 11

Today: viscous hydro from kinetic theory

in linear response (infinitesimally close to thermal eq.)

Parameterize: $f(x,p) = \exp\left[-\beta\left(\varepsilon(p) - \mu_{th} - \Phi(x,p)\right)\right]$

$$\approx f_{eq}(p) - \frac{\partial f_{eq}}{\partial \varepsilon}\Phi + \cdots \quad f_{eq}(1+\beta\Phi)$$

Similar ansatz for quantum Boltzmann ... [Take $H_1 = \varepsilon(p)$]

Plug into Boltzmann equation & keep linear order in Φ:

$$\partial_t(\beta f_{eq}\Phi) + \frac{\partial\varepsilon}{\partial p_i}\frac{\partial}{\partial x_i}(\beta f_{eq}\Phi) = \int d^d p_2\, d^d p_1'\, d^d p_2'\, R(p_1' p_2' \to p p_2) \times$$

$$\beta\left[\underbrace{f_{eq}(p_1')f_{eq}(p_2')}_{\text{equal}}\left(\Phi(p_1') + \Phi(p_2')\right) - \underline{f_{eq}(p)f_{eq}(p_2)}\left(\Phi(p) + \Phi(p_2)\right)\right]$$

this line equals

$$\beta f_{eq}(p_1')f_{eq}(p_2')\left[\Phi(p_1') + \Phi(p_2') - \Phi(p) - \Phi(p_2)\right]$$

Looks bad... "integro-differential equation":

Where does hydro come from?

$$\text{Idea: } \Phi(x,p) = \underbrace{\delta\mu(x) + \delta v_i(x)p_i - \frac{\delta\beta(x)}{\beta}\varepsilon(p)}_{\text{hydro modes}} + \overbrace{\delta c_{ij}(x)p_ip_j + \cdots}^{\text{non-hydro modes}}$$

Goal: show that Boltzmann "efficiently approximated" by hydro modes:

Because equations are linear in Φ... linear algebra!

Useful to think about:

$$\int d^dp\, \Psi(p)\, \beta f_{eq}^{(p)} \left[\partial_t \Phi + \frac{\partial \varepsilon}{\partial p_i}\frac{\partial \Phi}{\partial x_i} + \dots\right] = 0 \quad \text{for any } \Psi(p)$$

Define inner product: $\overset{\frac{\partial f_{eq}}{\partial \varepsilon}}{}$

$$\langle \Psi(p) | \Omega(p)\rangle = \langle \Psi | \Omega\rangle = \int d^dp\, \underbrace{\beta f_{eq}(p)}\, \Psi(p)\,\Omega(p)$$

$$\to \partial_t \langle \Psi | \Phi(x,t)\rangle + \frac{\partial}{\partial x_i}\langle \Psi | \frac{\partial \varepsilon}{\partial p_i} | \Phi\rangle + \underbrace{\langle \Psi | W | \Phi\rangle}_{W = \text{linearized collision integral}} = 0$$

$$\frac{\partial \varepsilon}{\partial p_i}|\Phi\rangle = |\frac{\partial \varepsilon}{\partial p_i}\Phi\rangle$$

W is positive semidefinite & (Lec 22) if [inv + time-rev.] - symmetry:

$$\downarrow$$
$$W = W^T$$

$$\langle \Phi | W | \Phi\rangle = \int d^dp_1 d^dp_2 d^dp_1' d^dp_2'\, R(p_1'p_2' \to p_1, p_2)\frac{\beta f_{eq}(p_1') f_{eq}(p_2')}{4}\left[\Phi(p_1') + \Phi(p_2') - \Phi(p_1) - \Phi(p_2)\right]^2$$

To reproduce old: $\frac{\delta}{\delta \Psi(p)}\left[\partial_t \langle \Psi | \Phi\rangle + \dots \langle \Psi | W | \Phi\rangle = 0\right]$

Zero modes / null vectors of W are associated w/ conserved q's.

$$\Phi(p_1') + \Phi(p_2') = \Phi(p_1) + \Phi(p_2) \quad \text{if } R(p_1'p_2' \to p_1 p_2) \neq 0.$$

Recap: $\overset{-i\omega}{\cancel{\partial_t}}|\Phi\rangle + ik_i V_i |\Phi\rangle + W|\Phi\rangle = 0$

$\hookrightarrow V_i = \frac{\partial \varepsilon}{\partial p_i}$

QNormal modes of kinetic theory are eigenvectors...

$\hookrightarrow$ hydrodynamic modes: $\omega(k) \to 0$ as $k \to 0$.

Can we classify these?

Let's start w/ $k_i = 0$: $\quad -i\omega|\Phi\rangle = W|\Phi\rangle$.

hydro mode $\longleftrightarrow$ null vector of W.

Since W is positive semidefinite:

$\omega = 0$ $\qquad$ hydro modes (slow)

$$W = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ & 0 & & & & \\ & & \ddots\, 0 & & & \\ \hline & & & 1/\tau_1 & & \\ & & & & 1/\tau_2 & \\ & & & & & \ddots \end{array}\right)$$

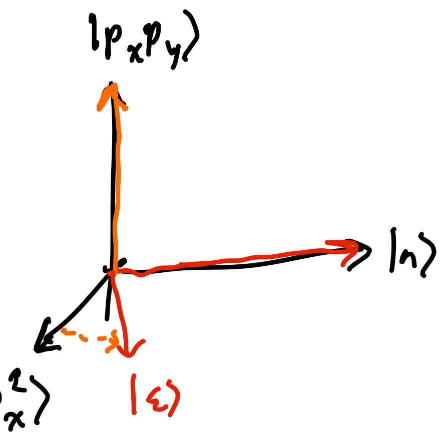
non-hydro (fast)

For ordinary (Galilean-invariant) gas: $d+2$ null vectors

$\lvert n\rangle$	$\Phi = 1$	(particle number)
$\lvert p_i\rangle$	$\Phi = p_i$	(momentum)
$\lvert \varepsilon\rangle$	$\Phi = \varepsilon(p) = p^2/2m$	(energy)

What happens if $k \neq 0$:

$$|\Phi\rangle = \begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} \begin{matrix} \text{hydro} \\ \text{not hydro} \end{matrix}$$



$$\partial_t|\Phi\rangle + ik\cdot V|\Phi\rangle + W|\Phi\rangle = 0$$

abstract $\hookrightarrow$
$$-i\omega\begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} + i\vec{k}\cdot\begin{pmatrix} V_{ss} & V_{sf} \\ V_{fs} & V_{ff} \end{pmatrix}\begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & W_f \end{pmatrix}\begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix}$$

Take hydrodynamic limit $\omega, k \to 0$:

$$\underbrace{\cancel{-i\omega|\Phi_f\rangle}}_{\omega \ll 1/\tau} + i\vec{k}\cdot V_{fs}|\Phi_s\rangle + \underbrace{\cancel{i\vec{k}\cdot\vec{V}_{ff}|\Phi_f\rangle}}_{kV_{typ} \ll 1/\tau} + \underbrace{W_f|\Phi_f\rangle}_{\hookrightarrow W_f^{-1}\text{ exists}} = 0.$$

so $|\Phi_f\rangle \approx -iW_f^{-1}\vec{k}\cdot\vec{V}_{fs}|\Phi_s\rangle \sim \mathcal{O}(k)$ out of equilibrium.

$$\left[-i\omega + i\vec{k}\cdot\vec{V}_{ss} + i\vec{k}\cdot\vec{V}_{sf}\left(-iW_f^{-1}\vec{k}\cdot\vec{V}_{fs}\right)\right]|\Phi_s\rangle = 0.$$

$$\left[-i\omega + i\vec{k}\cdot\vec{V}_{ss} + W'\right]|\Phi_s\rangle = 0$$

where effective collision integral $W' = k_i k_j (V_i)_{sf} W_f^{-1} (V_j)_{fs} \sim k^2$

Finite-dimensional problem for hydro modes!

Let's explicitly calculate V_{ss} & W' for ideal gas:

Recall: $|\Phi_s\rangle = \begin{pmatrix} |n\rangle \\ |p_i\rangle \\ |\varepsilon) \end{pmatrix}$ $\quad \begin{matrix} (1) \\ (p_i) \\ (p^2/2m) \end{matrix}$

$$\begin{pmatrix} \langle n| \\ \langle p_i| \\ \langle \varepsilon| \end{pmatrix}\left(-i\omega + \underline{i\vec{k}\cdot\vec{V}_{ss} + W'}\right)\left[\delta\mu|n\rangle + \delta v_i|p\rangle + \frac{\delta T}{T}|\varepsilon)\right] = 0.$$

Notice: $\langle n|\Phi\rangle = \delta\mu\langle n|n\rangle + \delta v_i\langle n|p_i\rangle + \cdots$

$$= \int d^d p\ \beta f_{eq}\cdot 1\cdot \Phi(p) = \int d^d p\ (f - f_{eq}) = \delta n$$

similarly $\langle p_i|\Phi\rangle = \delta g_i \qquad \langle \varepsilon|\Phi\rangle = \delta\varepsilon$

$\langle n|-i\omega|\Phi_s) = -i\omega\delta_n \sim \partial_t\delta n + \partial_i\delta J_i$

Inner products: $\langle n|n\rangle = \left.\frac{\partial n}{\partial\mu_{th}}\right|_{eq} = \chi_{nn}$ (thermodynamic susceptibility)

$$\langle n|\varepsilon) = \chi_{n\varepsilon} = \langle 1|\frac{p^2}{2m}\rangle = \int d^d p\ \frac{p^2}{2m}\beta e^{-\beta\left(\frac{p^2}{2m}-\mu_{th}\right)}$$

$$= \frac{\beta}{2m}e^{\beta\mu_{th}}\left(\frac{2\pi m}{\beta}\right)^{d/2}\cdot\frac{m}{\beta}d = \frac{d}{2}n \neq 0.$$

Our basis for slow modes NOT orthogonal.

$\vec{V}|n\rangle = \frac{\vec{p}}{m}|n\rangle = \frac{1}{m}|p_i\rangle$. So $V_{ss}|n\rangle = \frac{1}{m}|p_i)$ & $V_{fs}|n\rangle = 0.$

$V_{fs}|n\rangle = 0$ from Galilean boost sym... $\delta J_i = \delta v_j \underbrace{\langle p_j | \tfrac{1}{m} | p_i \rangle}$

$= \delta_{ij}\, n$

continuity: $-i\omega \delta n + i k_i n \delta v_i = 0$ $\quad [\partial_t \delta n + \partial_i (n \delta v_i) = 0]$

$$V_i |p_j\rangle = \frac{1}{m}|p_i p_j\rangle = \frac{1}{m}\Big[\underbrace{|p_i p_j - \tfrac{1}{d}\delta_{ij} p^2\rangle}_{\text{fast}}^{V_{fs}} + \underbrace{\frac{\delta_{ij}}{d}|p^2\rangle}_{\text{slow}}^{V_{ss} \to 2m|\varepsilon)}\Big]$$

rotation invariance $\leftarrow$ fast

"spin 2" irrep $\to$ traceless / symmetric

$$\langle p_j | [-i\omega + i k_i (V_i)_{ss}] | \Phi_s\rangle = -i\omega\, mn\, \delta v_j + \underbrace{\frac{i k_j}{d}\, 2\, \langle \varepsilon | \Phi_s\rangle}$$

$= i k_j [n \delta\mu + s \delta T] \to i k_j \delta P$

using thermo from Lec 8.

Full Navier-Stokes w/ dissipation...:

by symmetry

$$\langle p_\ell | W' | \Phi_s\rangle = ? = k_j k_k \langle p_\ell | (V_k)_{sf} (W_f^{-1}) (V_j)_{fs} \big[|p_i\rangle \delta v_i + \overset{0}{\cdots}\big]$$

$$= k_j k_k \underbrace{\left\langle \frac{p_k p_\ell - 1/d\, \delta_{k\ell} p^2}{m} \right| W_f^{-1} \left| \frac{p_i p_j - 1/d\, \delta_{ij} p^2}{m} \right\rangle}_{\eta_{k\ell j i}} \delta v_i$$

(viscosity tensor)

$-i\omega \delta g_\ell + i k_\ell \delta P + k_j k_k \eta_{k\ell j i} \delta v_i = 0$ is linearized N-S.

Notice $\eta_{k\ell j i} = \eta\left(\delta_{kj}\delta_{\ell i} + \delta_{ki}\delta_{\ell j} - \frac{2}{d}\delta_{k\ell}\delta_{ij}\right) + 0 \cdot \delta_{ij}\delta_{k\ell}$

by rot. sym $\quad\quad \hookrightarrow$ bulk viscosity

Predict: classical gas (air) have bulk viscosity $\zeta = 0$.

Often invokes relaxation time approximation:

$$W_f^{-1} \approx \tau$$

$$\eta = \frac{\langle p_x p_y | W_f^{-1} | p_x p_y \rangle}{m^2} = \frac{\tau}{m^2} \int d^d p \; \beta f_{eq} \, p_x^2 p_y^2$$

$$= \frac{\beta \tau}{m^2} \left(\frac{m}{\beta} \right)^2 \cdot n = \tau n T = \tau P$$

mean free time (τ); pressure (P)

If $\tau \to \infty$, then $\eta \to \infty$