PHYS 7810
Hydrodynamics
Spring 2024

Lecture 24
Ballistic-to-viscous crossover

April 16

Linearized Boltzmann equation $[-i\omega + i\vec{k}\cdot\vec{V} + W]|\Phi\rangle = 0$

linearized collision integral

$f = f_{eq} - \frac{\partial f_{eq}}{\partial \varepsilon}\Phi$

$f_{eq} = e^{-\beta(\varepsilon - \mu_\alpha)}$

Inner product: $\langle \Psi|\Phi\rangle = \int d^d p \left(-\frac{\partial f_{eq}}{\partial \varepsilon}\right)\Psi\Phi$

If $|\Phi\rangle = \delta\mu_\alpha |n^\alpha\rangle = \delta\mu|n\rangle + \delta v_i|p_i\rangle + \frac{\delta T}{T}|\varepsilon\rangle$ (slow modes)

and P_s projects onto slow modes... $P_s|\Phi\rangle = \delta\mu_\alpha|n^\alpha\rangle$:

arbitrary

$$P_s[-i\omega + i\vec{k}\cdot\vec{V} + W]|\Phi\rangle = 0$$

$$\hookrightarrow [-i\omega + i\vec{k}\cdot\underbrace{P_s\vec{V}P_s}_{V_{ss}} + k_j \underbrace{P_s V_i (1-P_s)}_{V_{sf}} W^{-1} \underbrace{(1-P_s)V_j P_s}_{V_{fs}} k_j\,]|n^\alpha\rangle\delta\mu_\alpha = 0.$$

thermodynamic/ ideal fluid

W' dissipative (e.g. viscosity)

Problem — how do we calculate viscosity? evaluate W^{-1}

Solution: (Chapman-Enskog) variational principle. If $M = M^T$

Claim: $\langle \Psi | M^{-1} | \Psi \rangle = \max_{|\Phi\rangle} \frac{\langle \Phi | \Psi \rangle^2}{\langle \Phi | M | \Phi \rangle} \rightsquigarrow R$

Proof (sketch): $\frac{\partial R}{\partial \Phi_\alpha} = \frac{2\langle \Phi | \Psi \rangle \Psi_\alpha}{\langle \Phi | M | \Phi \rangle} - \frac{2 M_{\alpha\beta}\Phi_\beta \langle \Phi | \Psi \rangle^2}{\langle \Phi | M | \Phi \rangle^2} = 0.$

Thus $|\Psi\rangle = \# M |\Phi\rangle$ on solution or $|\Phi\rangle = M^{-1}|\Psi\rangle$ ✓

$\hookrightarrow$ $R[\lambda|\Phi\rangle] = R[|\Phi\rangle]$ if $\lambda \neq 0$.

Idea: pick convenient basis for momentum space functions:

$$|\Phi\rangle = \underbrace{\delta\mu |1\rangle}_{|n\rangle} + \delta v_i |p_i\rangle + \underbrace{\frac{\delta T}{2mT}|p^2\rangle}_{|\varepsilon\rangle} + \delta c_{ij}\left(|p_i p_j - \tfrac{1}{d}p^2\delta_{ij}\rangle\right) + \cdots$$

$\nwarrow |\Phi_\alpha\rangle$

... truncate this expansion. Numerically evaluate $\langle \Phi_\alpha | M | \Phi_\beta \rangle$, $\langle \Phi_\alpha | \Psi \rangle$ for each basis function $|\Phi_\alpha\rangle$

... or just taking one basis function:

$$\langle \Psi | M^{-1} | \Psi \rangle \geq \frac{\langle \Phi | \Psi \rangle^2}{\langle \Phi | M | \Phi \rangle}.$$

Example: (HW6) shear viscosity: η_{xyxy}

From Boltzmann: $[-i\omega + i\vec{k}\cdot V_{ss} + W']|\Phi_s\rangle = 0$ $\rightarrow$ linearized Navier-Stokes...

Viscosity comes from: $\langle p_i|$: $-i\omega\,\delta g_i + \cdots + \eta_{jikl} k_j k_k \delta v_l = 0.$

$$\eta_{jikl} = \langle p_i | (V_j)_{sf} W^{-1} (V_k)_{fs} | p_l \rangle$$

Define $|\psi_{ji}\rangle = (V_j)_{fs}|p_i\rangle$.

$$\eta_{xyxy} \geq \frac{\langle \Phi | \psi_{xy}\rangle^2}{\langle \Phi | W | \Phi \rangle} \quad \text{or} \geq \frac{\langle \psi_{xy} | \psi_{xy}\rangle^2}{\langle \psi_{xy} | W | \psi_{xy}\rangle}$$

For Galilean-invariant gas: $|\psi_{xy}\rangle = \frac{1}{m}|p_x p_y\rangle$

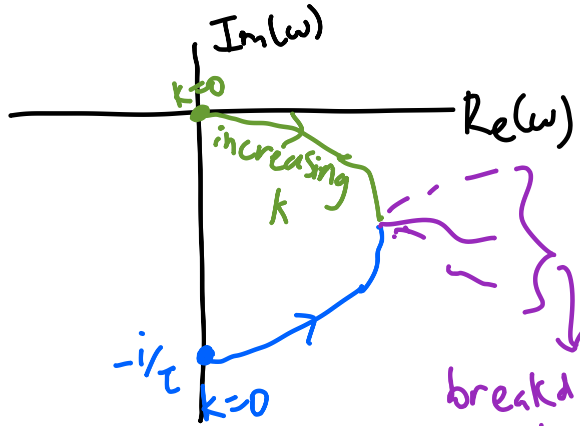
Let's define $\tau_\eta = \dfrac{\eta_{xyxy}}{\langle \psi_{xy} | \psi_{xy} \rangle} = \dfrac{\langle \psi_{xy} | W^{-1} | \psi_{xy} \rangle}{\langle \psi_{xy} | \psi_{xy} \rangle}$

Postulate: if W's eigenvalues "were all similar", then just approximate $W \approx \frac{1}{\tau_\eta}(1-P_s)$, $\longleftrightarrow$ "relaxation time approximation"

$\rightarrow$ useful cartoon for beyond hydro regime!

Recall Lec 7:

$\omega =$ pole in $\langle \rho\rho \rangle$



sound mode (hydro)

not hydro

breakdown of hydrodynamics at large freq. / wave number

Access these non-hydro modes using kinetic theory?

$\hookrightarrow$ ballistic regime = $\omega \gtrsim 1/\tau$, dominant terms are streaming

contrast w/ hydro regime: $\omega \ll 1/\tau$.

Want a model that captures both.

Minimal toy model for viscous $\rightarrow$ ballistic crossover?

Relativistic 1d gas (massless particles)

Hydro modes: $|n\rangle$ $\quad |p_x\rangle \quad |\varepsilon\rangle$

$\Phi = 1 \quad \Phi = p \quad \Phi = c|p|$

Not hydro: everything else

Full Boltzmann: $\left[\partial_t + ikV + \frac{1}{\tau}(1-P_s)\right]|\Phi\rangle = 0.$

$\frac{1}{\tau} \rightarrow \gamma$

$V|p_0\rangle = \begin{cases} +c & p_0 > 0 \\ -c & p_0 < 0 \end{cases} \cdot |p_0\rangle$

$\Phi(p) = \delta(p - p_0)$

Suppose an initial condition (@ fixed wave number) $|\Phi(t=0)\rangle = |\Phi_0\rangle$.
How to calculate $|\Phi(t)\rangle$?

Laplace transform: $|\Phi(z)\rangle = \int_0^\infty dt\, e^{-zt} |\Phi(t)\rangle$

$$[ikV + \gamma(\mathbb{1} - P_s)]|\Phi(z)\rangle + z|\Phi(z)\rangle - |\Phi(0)\rangle = 0.$$

or $$|\Phi(z)\rangle = [z\mathbb{1} + ikV + \gamma(\mathbb{1} - P_s)]^{-1}|\Phi(0)\rangle.$$

Initial perturb: $|\Phi_0\rangle \propto |n\rangle$ ... what is $\delta n(x,t) \sim e^{ikx}\delta n(t)$

$$\delta n(z) = \underbrace{\langle n|\Phi(z)\rangle}_{G(z)} = \langle n|[z\mathbb{1} + ikV + \gamma(\mathbb{1} - P_s)]^{-1}|n\rangle$$

$G(z) \to$ Green's function

$$G(z) = \langle n|[\underbrace{z + \gamma + ikV}_{\text{call } G_0^{-1}} - \gamma P_s]^{-1}|n\rangle$$

$$G_0|p_0\rangle = \frac{1}{z + \gamma + ik(\underset{\text{sign}(p_0)}{\pm c})}|p_0\rangle$$

$$= \langle n|[G_0^{-1} - \gamma P_s]^{-1}|n\rangle = \langle n|\underset{\hat{P}_s}{G_0} + G_0\gamma P_s G_0 + G_0\gamma P_s G_0 \gamma P_s G_0 + \cdots \underset{\hat{P}_s}{|n\rangle}$$

If $\tilde{G} = P_s G_0 P_s$, then $$G(z) = \langle n|\tilde{G} + \gamma\tilde{G}^2 + \cdots|n\rangle$$
$$= \langle n|\tilde{G}(\mathbb{1} - \gamma\tilde{G})^{-1}|n\rangle$$

$\tilde{G}$ is a 3×3 matrix (slow modes: $|n\rangle$, $|p_x\rangle$, $|\varepsilon\rangle$).
Finite-dimensional problem has exact solution!

For projection in P_s ... help to have orthonormal basis:

$$|1\rangle = \frac{|n\rangle}{\sqrt{\langle n|n\rangle}} \qquad |2\rangle = \frac{|p_x\rangle}{\sqrt{\langle p_x|p_x\rangle}} \qquad |3\rangle = \frac{|\tilde{\varepsilon}\rangle}{\sqrt{\langle\tilde{\varepsilon}|\tilde{\varepsilon}\rangle}}$$

$$|\tilde{\varepsilon}\rangle = |\varepsilon\rangle - |1\rangle\langle 1|\varepsilon\rangle$$
$$= |\varepsilon\rangle - |n\rangle\frac{\langle n|\varepsilon\rangle}{\langle n|n\rangle}$$

$$\langle n|n\rangle = \chi_{nn} = \int_{-\infty}^{\infty} dp\ \beta e^{-\beta(c|p|-\cancel{\mu})} \xrightarrow{} 0 \text{ to reduce clutter} = \int_0^\infty dp\cdot 2\beta e^{-\beta cp} = \frac{2}{c}$$

$$\langle p|p\rangle = \int_{-\infty}^{\infty} dp\ \beta e^{-\beta c|p|}\, p^2 = \frac{4}{\beta^2 c^3} = \langle \varepsilon|\varepsilon\rangle$$

$$\langle n|\varepsilon\rangle = \int_{-\infty}^{\infty} dp\ e^{-\beta c|p|}\beta\cdot|p| = \frac{2}{\beta c^2}$$

$$\Big\}\ \langle\tilde{\varepsilon}|\tilde{\varepsilon}\rangle = \frac{2}{\beta^2 c^3}$$

$$\langle 1|\tilde{G}|1\rangle = \frac{c}{2}\langle n|\tilde{G}|n\rangle = \frac{c}{2}\int_{-\infty}^{\infty} dp\ \beta e^{-c\beta|p|}\frac{1}{z+\gamma+ikc\cdot \mathrm{sign}(p)}$$

$$= \beta c\int_0^\infty dp\ e^{-\beta cp}\frac{z+\gamma}{(z+\gamma)^2+c^2k^2} = \frac{z+\gamma}{(z+\gamma)^2+(ck)^2}$$

Keep calculating:

$$\tilde{G} = \frac{1}{(z+\gamma)^2+(ck)^2}\begin{pmatrix} z+\gamma & ick/\sqrt{2} & 0 \\ ick/\sqrt{2} & z+\gamma & ick/\sqrt{2} \\ 0 & ick/\sqrt{2} & z+\gamma \end{pmatrix}$$

Now we need $G(z) = \langle 1|\tilde{G}(\mathbb{1}-\gamma\tilde{G})^{-1}|1\rangle$

$$= \frac{2z(c^2k^2+z^2)+\gamma(c^2k^2+2z^2)}{2(c^2k^2+z^2)(c^2k^2+z(\gamma+z))}$$

Laplace ~ Fourier $(z \rightsquigarrow -i\omega)$:

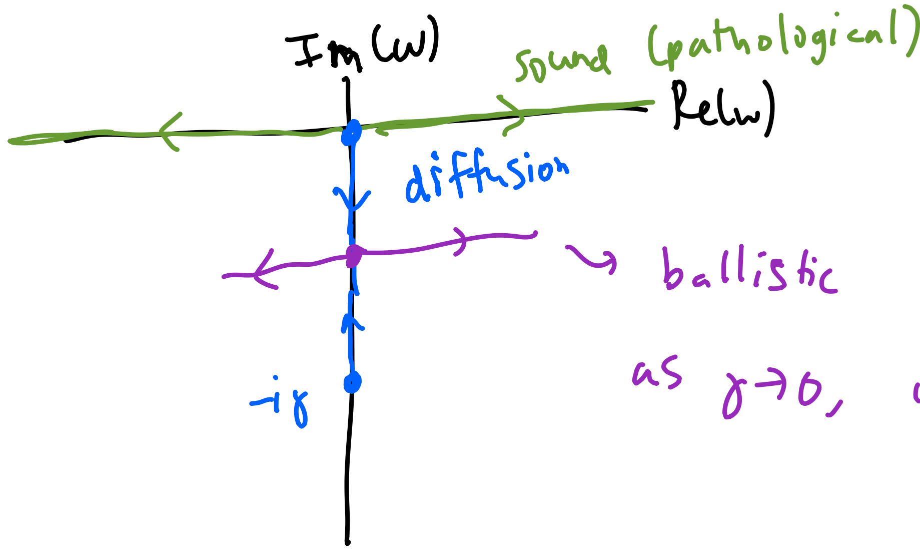
$$G(z) = \frac{\text{numerator}}{\underbrace{(c^2k^2-\omega^2)}\underbrace{(c^2k^2-i\omega(\gamma-i\omega))}}$$

Structure in correlation function $\longrightarrow$ poles/branch cuts of $G(z)$

Sound: $\omega = \pm ck$ $\quad$ ($|\varepsilon\rangle$ & $|p_x\rangle$ form "closed" subset)

$$\omega = -i\,\frac{\gamma\pm\sqrt{\gamma^2-(2ck)^2}}{2} \longrightarrow \begin{cases} -i\,\frac{c^2}{\gamma}k^2 & \text{(diffusion)} \\ -i\gamma & \text{(not hydro)} \end{cases} \quad \text{as } k\to 0$$

Correlation function sketch:



ballistic $2c|k| > \gamma$

as $\gamma \to 0$, $\omega \approx \pm ck$ (ballistic motion w/ speed c)