PHYS 7810
Hydrodynamics
Spring 2024

Lecture 25
Superfluids

April 18

Today: hydro w/ spontaneous symmetry breaking (SSB)

↓ What is this?

One example (lec 27): a solid.

translating solid generates new soln of EOMs...

translation symmetry of EOMs... but not state.

Minimal example of SSB: superfluid [break U(1) symmetry]

Consider "microscopic" Lagrangian:

$$\mathcal{L} = -\hbar\bar{\psi}\, i\partial_t \psi + \frac{\hbar^2}{2m}\partial_i\bar{\psi}\,\partial_i\psi - V(\underbrace{|\psi|^2}_{=\bar{\psi}\psi})$$

U(1) symmetry:

$$\psi \to \psi e^{+i\theta} \qquad \bar{\psi} \to \bar{\psi} e^{-i\theta} \qquad (\text{here } \theta = \text{const.})$$

By Noether's Theorem: there exists conserved current:

$$\underset{\substack{\uparrow \\ \mu=(t,i)}}{J^\mu} = -i\psi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} + i\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} \quad : \quad \partial_\mu J^\mu = \partial_t J^t + \partial_i J^i = 0$$

$J^t = \rho = -i\psi(-i\bar{\psi}) = -\bar{\psi}\psi$
(charge density)

SSB: V has minimum at $|\psi|^2 = \psi_0^2 > 0$.
"ground states": $\psi = \psi_0 e^{+i\theta}$ — θ: tunable const.

Consider a hydrodynamic description...
I have conserved J^μ. Can J^i be $f(\rho, \partial_i\rho \ldots)$? Answer later...

Suppose $\psi = \sqrt{\rho(x)}\, e^{i\theta(x)}$, $\bar{\psi}(x) = \sqrt{\rho}\, e^{-i\theta}$

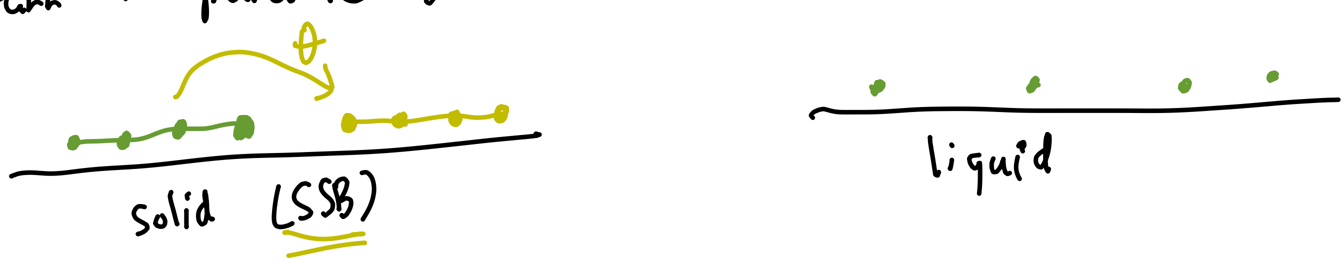
$$\hookrightarrow \mathcal{L} = \cancel{\hbar}\rho\,\partial_t\theta \underbrace{- \frac{\rho^2}{2m}(\nabla\theta)^2 + \cdots - V(\rho)}_{-\mathcal{H}(\rho,\, \nabla\theta,\, \nabla\rho,\, \ldots)}$$

Analogy: $L = p\dot{x} - H(x,p) \longrightarrow$ Hamiltonian mechanics: $\{x,p\} = 1$.

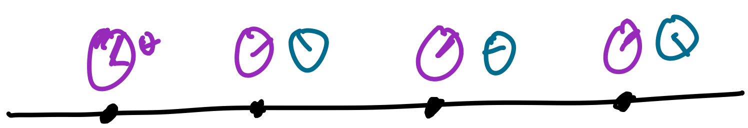
$\hookrightarrow \{\rho(x), \theta(y)\} = \delta(x-y)$ at classical level

Key: hydro EFT for superfluid involve both ρ & θ.
ρ & θ are canonical conjugate DOF.

Return to parable of solid:



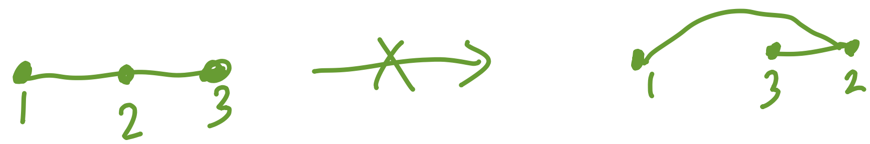
θ's get "locked" in superfluid $\longrightarrow$ Goldstone boson
without SSB



In symmetry-preserving $U(1)$: tune phase separately at each point

↳ hydro should not involve θ.

In SSB/superfluid: Goldstone for $\theta(x)$ dynamics is a hydro DOF.



Propose: normal fluid (no SSB): $H \to H(\rho)$

superfluid (SSB) : $H \to H(\underbrace{\partial_i\theta}, \rho)$

$\theta \to \theta + \text{const.}$ is $U(1)$ symmetry

MSR Lagrangian for normal fluid: (cf lec 4-5)

Suppose our steady state $P_{ss} = e^{-\Phi} = e^{-\beta H}$

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_\theta \partial_t \theta - T[\pi_\rho \underbrace{\mu_\theta}_{=\frac{\delta H}{\delta \theta}} - \pi_\theta \mu_\rho] + i\partial_i \pi_\rho (\sigma T)\partial_i(\pi_\rho - i\mu_\rho) + \cdots$$

$\partial_i \pi_\rho$ allowed since $\pi_\rho \to \pi_\rho + 1$ needed due to charge conservation

If $H = H(\rho)$ ... $\underline{\mu_\theta = 0.}$

Using $\mu_\rho = \beta \frac{\delta H}{\delta \rho} \simeq \frac{\rho}{\chi}$ if $H = \int d^dx \, \frac{\rho^2}{2\chi} + \cdots$

equations of motion:

$$\partial_t \rho - \partial_i\left(\sigma\, \partial_i \frac{\rho}{\chi}\right) + \cdots = 0$$

← diffusion equation of normal fluid.

$$\partial_t \theta + \frac{\delta H}{\delta \rho} = \underbrace{\partial_t \theta + \frac{\rho}{\chi}}_{} \simeq 0, \qquad \frac{\rho}{\chi} \to \mu_{th}$$

Josephson relation: phase rotates at "const." speed $\propto$ chemical potential.

This equation doesn't matter for ordinary fluid.

MSR for superfluid:

new!

$$\beta H = \beta \int d^d x \left[\frac{\rho^2}{2\chi} + \boxed{\frac{K}{2}(\nabla\theta)^2} + \cdots \right]$$

Note: $\rho \to \rho - \rho_0$

same!

$$\mu_\rho = \beta \frac{\rho}{\chi} \qquad \mu_\theta = \beta K(-\nabla^2 \theta)$$

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_\theta \partial_t \theta - T(\pi_\rho \mu_\theta - \mu_\rho \pi_\theta) + i T\sigma \partial_i \pi_\rho \partial_i (\pi_\rho - i\mu_\rho) + \cdots$$

Equations of motion:

$$0 = \partial_t \theta + \rho/\chi \quad \text{(Josephson)}$$

$$0 = \partial_t \rho + K \partial_i \partial_i \theta - \partial_i\left(\sigma \partial_i \frac{\rho}{\chi}\right) \quad \text{(charge conservation)}$$

Assume θ, ρ small:

$$0 = -\chi \partial_t^2 \theta + K \partial_i \partial_i \theta + \sigma \partial_i \partial_i \partial_t \theta$$

Quasinormal modes: $\theta \sim e^{ikx - i\omega t}$:

$$v_s^2 = \frac{K}{\chi}, \quad D = \frac{\sigma}{\chi} \quad \ldots \quad \omega^2 - v_s^2 k^2 + i\omega k^2 D = 0$$

$$\omega \approx \pm v_s k - \frac{i}{2} D k^2 + \cdots$$

Now: ballistic mode. Goldstone boson

- approach normal fluid by $K \to 0$. ($\omega = 0$ and $\omega = -iDk^2$)
- superfluid dissipation allowed. $D > 0$ generally...

Often conventional to write

superfluid velocity $u_i = \frac{K}{\rho} \partial_i \theta \sim \frac{\hbar}{m} \partial_i \theta$

Hydro equations:

$$\partial_t \rho + \partial_i \left(\rho u_i - \sigma \partial_i \rho/\chi \right) = 0 \quad \text{(charge cons.)}$$

$$\partial_t (\rho u_i / K) + \partial_i \rho/\chi = 0 \quad \text{(Josephson)} \rightsquigarrow \text{N-S w/o viscosity.}$$

So far, neglected noise in θ. Does it matter?

Modify $\mathcal{L}_{MSR} \to \mathcal{L}_{MSR} + i\gamma T \pi_\theta (\pi_\theta - i\mu_\theta)$

Josephson relation: $\partial_t \theta + \rho/\chi = K_\gamma \nabla^2 \theta$

Plug in to ρ EOM:

$$K\nabla^2\theta + (\partial_t - D\nabla^2)(-\chi(\partial_t\theta - K_\gamma \nabla^2\theta)) = 0$$

$$\hookrightarrow v_s^2 \nabla^2\theta - \partial_t^2\theta + \partial_t \nabla^2\theta \cdot \underbrace{(D + K_\gamma)}_{D_{\mathrm{eff}} = D + K_\gamma} + \cancel{\mathcal{O}(\nabla^4)} = 0.$$

$\omega \sim k$

$D_{\mathrm{eff}} = D + K_\gamma$ ... same dispersion as before!

Illustration: fluid frame transformation... some change-of-vars are unphysical... $\rho \approx -\chi \partial_t \theta + \mathcal{O}(\nabla^2)$ counteracted by shifts in D, ...

Popular choice of fluid frame for ordinary fluid:

Landau frame: $J^t = \rho$ AND $\mu = f(\rho)$

Popular choice for superfluid:

drop all $\mathcal{O}(\pi_\theta^2)$ corrections...

makes Josephson $\partial_t \theta + T\mu_\rho = 0$ exact.

or $\partial_t \theta = -\mu_{th}$

$\hookrightarrow$ π_θ into Lagrange multiplier... $T\mu_\rho \to -\partial_t\theta$.

$$\mathcal{L}_{MSR} \to \underbrace{\mathcal{L}(\partial_t\theta, \nabla\theta, \pi_\rho)}_{\text{limit of normal fluid blocks } \nabla\theta\,?}$$

enforce by "reparameterization" symmetry $\theta \to \theta + f(\vec{x})$

arbitrary t-ind.

$\hookrightarrow$ ~2015

birth of modern hydro EFT: $\theta \to \theta_r$

$\pi_\rho \to \theta_a$