PHYS 7810
Hydrodynamics
Spring 2024

Lecture 26
Superfluids with energy and momentum

April 23

Recall: superfluid hydro has spontaneous sym. breaking

The Goldstone boson θ (conjugate to ρ) $\rightarrow$ slow DOF

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_\theta \partial_t \theta - T(\pi_\theta \mu_\rho - \pi_\rho \mu_\theta) + i\sigma T \partial_i \pi_\rho \partial_i (\pi_\rho - i\mu_\rho)$$

$\mu_\theta \to -K\nabla^2\theta$, $\quad \mu_\rho \to \rho/\chi$

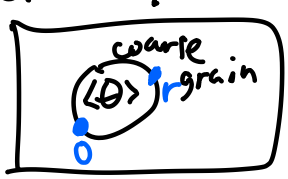
Since $\Phi = \beta H = \beta \int dx \left[\frac{\rho^2}{2\chi} + \frac{K}{2}(\nabla\theta)^2 \right]$

normal fluid $\quad \rightsquigarrow \quad$ superfluid

$\omega = -iDk^2 \qquad\qquad \omega = \pm v_0 k - i\frac{Dk^2}{2} + \ldots$

Claim: our SF hydro theory ill-posed in $d \le 2$ spatial dimensions

(Mermin-Wagner Thm)



Is $\langle\theta\rangle > 0$ in large domain (hydro) limit?

$\langle \theta(r)\, \theta(0)\rangle - \langle\theta\rangle^2 > 0$ as $r \to \infty$?

If true, system has long-range order.

$\rightarrow$ required for SF

Evaluate Gaussian path integral:

$$\langle \theta(r)\theta(0)\rangle - \langle\theta\rangle^2 = \int D\theta\, e^{-\beta H[\theta]}\theta(r)\theta(0) = \frac{1}{K}\underbrace{(-\nabla^2)^{-1}}(r-0)$$

Green's function G: $\nabla^2 G(x) = \delta(x)$

$$G(r) \sim \begin{cases} -|r| & d=1 \\ -\log r & d=2 \\ \boxed{\frac{1}{r^{d-2}} \quad d>2} \end{cases} + \underbrace{\text{const.}}_{\text{UV/small scale.}}$$

Superfluid phase exists $T>0$.

In $d \leq 2$, coarse-graining would replace $\Phi = \int d^dx\, K(\nabla\theta)^2$

$K_{eff} \to 0$

(no SSB at $K=0$)

In $d=2$: if $\theta \sim \theta + 2\pi$, there's a transition at $T = T_{BKT}$

to absence of topological defect (vortex)

$\hookrightarrow F_{vortex} \sim \Delta E_{vortex} - T\underbrace{\Delta S_{vortex}}_{\log(L^2)}$ ← system size

ΔE_{vortex}: $\int(\nabla\theta)^2 \sim \int \frac{d^2r}{r^2} \sim \log L$

Now: superfluid w/ energy & momentum conservation

(liquid He^4)

(schematic)

$$\mathcal{L}_{MSR} = \mathcal{L}^{normal}_{MSR} + \pi_\theta \partial_t\theta - T(\pi_\theta\mu_\rho - \pi_\rho\mu_\theta) + \cdots$$

$$\Phi = -\int d^dx\, s(\varepsilon, g_i, \rho, \partial_i\theta)$$

↳ shift $\theta \to \theta + c$ is broken symmetry

Hydro = gradient expansion: $\partial_t\rho + \partial_i J_i = 0$ ← keep few der. as possible

Treat $\partial_i\theta$ at same order as ε, g_i, ρ.
$\sim u_i$ (superfluid velocity)

Notice: under time-reversal T: $\rho \to \rho$ $\quad \pi_\rho \to -\pi_\rho + i\mu_\rho$
$\theta \to -\theta$ $\quad \pi_\theta \to \pi_\theta - i\mu_\theta$

Focus on Galilean boost invariance (liquid He^4):

$$\underset{\uparrow \text{const.}}{v_i \to v_i + c_i} \qquad \underbrace{\theta \to \theta + \frac{m}{\hbar} c_i x_i}$$

superfluid velocity: $u_i = \frac{\hbar}{m}\partial_i\theta \to u_i + c_i$

We expect (cf lec 9): $\quad w_i = u_i - v_i$

$$s = s\left(T, \mu_m + \frac{v^2}{2}, (u-v)^2\right)$$

skipping steps... constitutive relations (at ideal fluid level):

$$\partial_t \rho + \partial_i J_i = 0 \qquad J_i = \rho v_i + \rho_s w_i$$
$$\partial_t g_i + \partial_j \tau_{ji} = 0 \qquad \tau_{ji} = P\delta_{ij} + \rho v_i v_j + \rho_s w_i w_j$$
$$\partial_t \varepsilon + \partial_i \mathcal{E}_i = 0 \qquad \mathcal{E}_i = (\varepsilon + P) v_i + \mu \rho_s w_i + \cdots$$

and Josephson relation:

$$\partial_t u_i + v_j \partial_j u_i = \partial_i\left(\mu_m + \frac{v^2}{2}\right)$$

Superfluid density:

$$\rho_s = \frac{\partial P}{\partial \frac{w^2}{2}}$$

Useful interpretation (Landau–Tisza): two-fluid model

$$J_i = \rho v_i + \rho_s w_i = \underbrace{(\rho - \rho_s)}_{=\rho_n \text{ (normal fluid density)}} v_i + \rho_s u_i \leftarrow \text{SF}$$

Dissipative corrections mix 2 fluids... not 2 actual fluids...
↳ what are they? Focus on linear response around $v = w = 0$.

Look for T-invariant building blocks in $\mathcal{L}_{MSR}$

normal Fluid: $i\kappa T^2 \partial_i \pi_\varepsilon \partial_i (\pi_\varepsilon - i\mu_\varepsilon) + i \partial_i \pi_i T \zeta \partial_j (\pi_j - i\mu_j) + i\eta T \partial_i \pi_j \cdots$

thermal cond. bulk visc. shear visc.

Superfluid: $i\gamma \pi_\theta (\pi_\theta - i\mu_\theta) + i\zeta \pi_\theta (\partial_j (\pi_j - i\mu_j)) + i\alpha \partial_i \pi_i (\pi_\theta - i\mu_\theta)$

5 total dissipative coefficients

constraints: $\kappa \geq 0$, $\eta \geq 0$, $\gamma, \zeta \geq 0$, $\gamma\zeta \geq \alpha^2$

"Dissipationless superfluid flow... no shear viscosity"

Claim: SF (liquid He^4) has 2 sound modes:

$\delta\rho, \delta\varepsilon, \delta g_i, \delta u_i \sim e^{ikx - i\omega t}$

Neglect dissipation...

$-i\omega \delta\rho + ik(\rho\delta v_x + \rho_s \delta w_x) = 0$

$-i\omega \delta v_x \rho + ik\delta P \approx 0$

$\boxed{-i\omega\rho\delta v_\perp + \eta k^2 \delta v_\perp = 0}$ → transverse momentum decouples: $\omega = -i\nu k^2$ ← kinematic visc $\nu = \frac{\eta}{\rho}$.

$-i\omega\delta\varepsilon + ik(\varepsilon + P)\delta v_x + ik\mu\rho_s \delta w_x \approx 0$

$-i\omega \delta u_x + ik\delta\mu = 0$ (Josephson relation)

Remember: $u_i \sim \partial_i \theta \sim k_i \theta$ (no transverse u)

Note: $\omega = ck$, $c \sim$ speed(s) of sound.

Solve eigenvalue problem for c.

Relate $\delta\rho, \delta\varepsilon, \delta P, \delta\mu$ ... $\delta\rho, \delta s$ independent

$$i\omega \underbrace{(\delta\varepsilon - \mu\delta\rho)}_{T\delta s} = ik\underbrace{(\varepsilon + P - \mu\rho)}_{Ts}\delta v_x + \cancel{ik\mu\rho_s(\delta w_x - \delta w_x)}$$

And: $dP = \rho\, d\mu + s\, dT + O(w^2)$

Combine:

$$c \begin{pmatrix} \delta\rho \\ \delta s \\ \delta v_x \\ \delta u_x \end{pmatrix} = \begin{pmatrix} 0-c & 0 & \rho-\rho_s & \rho_s \\ 0 & 0-c & s & 0 \\ \frac{1}{\rho}\frac{\partial P}{\partial \rho} & \frac{1}{\rho}\frac{\partial P}{\partial s} & 0-c & 0 \\ \frac{1}{\rho}\left(\frac{\partial P}{\partial \rho} - s\frac{\partial T}{\partial \rho}\right) & \frac{1}{\rho}\left(\frac{\partial P}{\partial s} - s\frac{\partial T}{\partial s}\right) & 0 & 0-c \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta s \\ \delta v_x \\ \delta u_x \end{pmatrix}$$

Eigenvalues given by: $0 = \det([4\times 4] - c\mathbb{1})$

Find characteristic polynomial AND thermodynamic identities:

$$0 = c^4 - c^2\left[\left.\frac{\partial P}{\partial \rho}\right|_{\rho/s} + \frac{\rho_s T s^2}{\rho_n c_v}\right] + \frac{\rho_s T s^2}{\rho_n c_v}\left.\frac{\partial P}{\partial \rho}\right|_T$$

two (left/right) sound modes obey:

$$c^2 = \frac{1}{2}\left[\left.\frac{\partial P}{\partial \rho}\right|_{\rho/s} + \frac{\rho_s T s^2}{\rho_n c_v} \pm \underbrace{\sqrt{\left(\left.\frac{\partial P}{\partial \rho}\right|_{\rho/s} + \frac{\rho_s T s^2}{\rho_n c_v}\right)^2 - 4\,\frac{\rho_s T s^2}{\rho_n c_v}\left.\frac{\partial P}{\partial \rho}\right|_T}}_{\sqrt{\left(\frac{\partial P}{\partial \rho} - \frac{\rho_s T s^2}{\rho_n c_v}\right)^2}}\right]$$

Approximate: $\left.\frac{\partial P}{\partial \rho}\right|_{\rho/s} \approx \left.\frac{\partial P}{\partial \rho}\right|_T$

Then 2 solutions:

first sound (usual fluid sound) $\quad c_1^2 = \frac{\partial P}{\partial \rho}$ ← δP large, δT small $\quad \delta w_x \approx 0.$ $\delta v_x \approx \delta u_x$

Second sound "new" to SF $\quad c_2^2 = \frac{\rho_s T s^2}{\rho_n c_v}$ ← $c_2 \to 0$ as $\rho_s \to 0$. $(K \to 0)$

δT large, $\delta P \approx 0$ $\quad \rho_n \delta v_x + \rho_s \delta u_x \approx 0.$