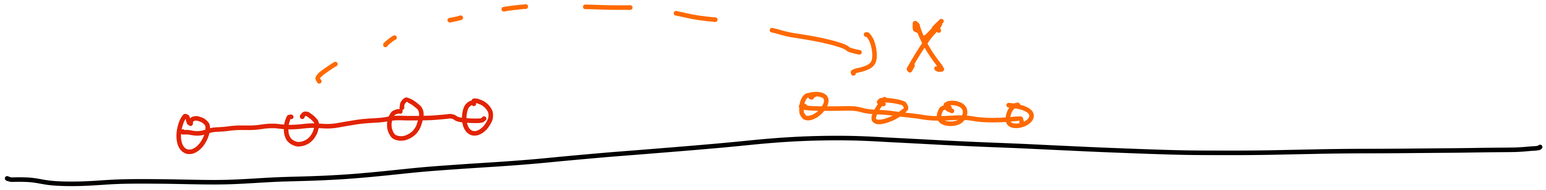
PHYS 7810
Hydrodynamics
Spring 2024

Lecture 27
Viscoelastic solids

April 25

Today: Solids are hydrodynamic w/ translation SSB



state is NOT invariant under translation symmetry.

For simplicity, neglect mass/energy conservation.

Noether's Thm: translation symmetry

$$\downarrow$$

momentum conservation

Lec 25: account for SSB by adding Goldstone boson as a hydro DOF.

$\downarrow$

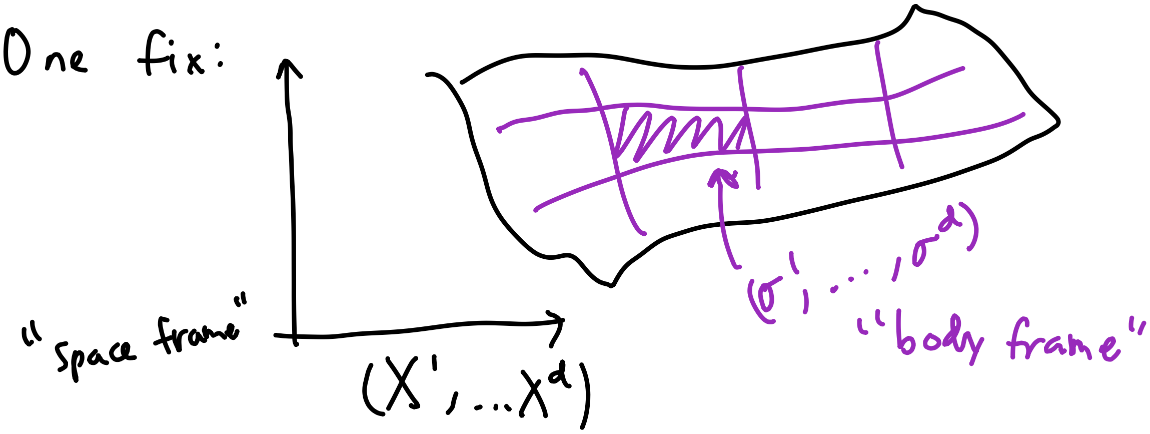
X_i = position of solid

We have canonical Poisson bracket:

$$\text{“} \{ X_i(x), g_j(x') \} = \delta_{ij}\, \delta(x-x') \text{”}$$

Awkward b/c... X is also a coordinate?

One fix:



fields: $X_i(\sigma^I)$

$\downarrow$

Lagrangian description.

Complication:

$$\frac{d}{dt}\int d^dX \; g_i(X) = 0 \qquad \text{NOT} \qquad \frac{d}{dt}\int d^d\sigma \; g_i(\sigma) = 0.$$

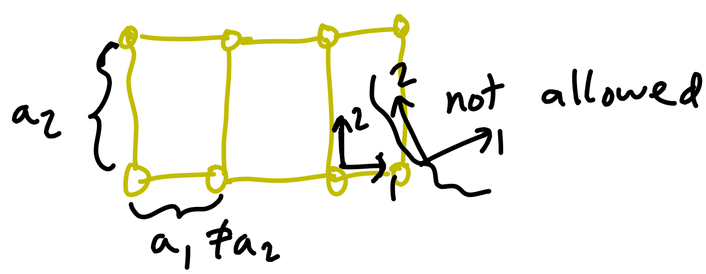
Alternative possibility: Eulerian description: $\sigma^I(X_i)$

Technical: this requires $\sigma(X)$ invertible.

Neglect since focus on long wavelength

Before MSR: what are symmetries?

- translation symmetry: $X_i \to X_i + c_i$
- space frame rotation: $X_i \to R_{ij} X_j$ ← orthogonal
- $\sigma^I \to \sigma^I + c$ ("relabeling")
- NOT have σ^I rotation sym!



Important: anisotropic crystals ...

... $\approx$ all crystals in 3d

Next: building Φ (steady state)

Crude: $\Phi = \beta \int d^dX \, \underbrace{\frac{\kappa_{ij}}{2} g_i g_j}_{\text{"}\sum p^2/2m\text{"}} + \cdots \underbrace{\Phi_{el}[X]}$ (analogy to lec 25)

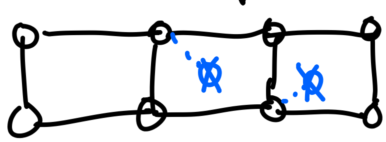
translation: $\Phi_{el}[X] = \Phi_{el}[X+c]$

or $\Phi_{el} = \int d^dX \, f(\underbrace{\partial^I X_i}, \dots)$

use Lagrangian desc.

What's σ? Make a useful choice:

Solid have preferred equilibria:



$\sigma^I = \delta^I_i X_i$ + const.

↳ labels each cell's position "at rest"

Notice: $\partial^I X_i$ should be orthogonal: $\partial^I X_i \partial^J X_i = \delta^{IJ}$.

Now postulate: "elastic moduli" encode crystal's symmetry group

$$\Phi_{el} = \beta \int \underbrace{d^dX}_{\hookrightarrow\, d^d\sigma \, \det\left(\frac{\partial X}{\partial \sigma}\right) \approx 1} \frac{1}{8} \lambda^{IJKL} \left(\partial^I X_i \partial^J X_i - \delta^{IJ}\right)\left(\partial^K X_j \partial^L X_j - \delta^{KL}\right)$$

$\lambda^{(IJ)(KL)}$ should be positive semidefinite $\rightsquigarrow$ thermodynamic stability.

Note: $\kappa_{ij} \to \partial^I X_i \partial^J X_j \underbrace{\kappa^{IJ}}_{\text{anisotropic?}}$

Focus on small perturbations near equilibrium

$\sigma^I = \delta^I_i (X_i + \phi_i(\sigma))$

↳ $\approx \phi_i(X)$ to lowest order in ϕ.

Lazy: (in linear response / leading order in ϕ):

$$\{\phi_i(x), g_j(x')\} = \delta_{ij}\,\delta(x-x')$$

Hydro EFT: $\mathcal{L}_{MSR}(\phi_i, \pi_{\phi_i}, g_i, \pi_{g_i})$, $\pi_{\phi_i} \downarrow \pi_i$, $\pi_{g_i} \downarrow \psi_i$

Under time-reversal: $g \to -g$ $\qquad \phi \to \phi$

$$\psi_i \to \psi_i - i\frac{\delta\Phi}{\delta g_i} \qquad \pi \to -\pi + i\mu_\phi \qquad \mu_i = \frac{\delta\Phi}{\delta\phi_i}$$

$$\psi_i - i\nu_i, \qquad \nu_i = \beta v_i$$

$$\mathcal{L}_{MSR} = \pi_i \partial_t\phi_i + \psi_i\partial_t g_i - T(\pi_i v_i - \psi_i\mu_i) + i\eta_{ijkl}T\partial_i\psi_j\,\partial_k(\psi_l - i\nu_l)$$

dissipate (viscous)

momentum conservation: $\psi_i \to \psi_i + c_i$

$$\nu_i = \frac{1}{T}v_i = \frac{1}{T}K_{ij}g_j$$

$$\mu_i = \frac{\delta\Phi_{el}}{\delta\phi_i} \quad \ldots \quad \frac{\delta\Phi}{\delta X_i} \approx -\frac{\beta}{4}(\lambda^{IJKL} + \lambda^{JIKL})\,\partial^I\partial^J X_i(\partial^K X_j\partial^L X_j - \delta^{KL})$$

$$\approx -\frac{\beta}{4}(\lambda^{IJKL} + \lambda^{JIKL})\,\partial^I\delta_i^J(\partial^L\phi^K + \partial^K\phi^L)$$

$$\to -\beta\underbrace{\lambda_{ijkl}}\partial_j\partial_k\phi_l \qquad \text{(space \& body frame locked)}$$

$$\lambda_{ijkl} = \lambda_{klij} = \lambda_{jikl}$$

So the EOMs: (neglect noise)

(momentum cons.) $\partial_t g_i - \eta_{ijkl}\partial_j\partial_k v_l - \lambda_{ijkl}\partial_j\partial_k\phi_l = 0$

(Josephson rel.) $\partial_t \phi_i - v_i = 0$

$\hookrightarrow = \mathcal{K}_{ij} g_j$

Combine: $\underbrace{\text{visco}}\underbrace{\text{elastic}}$ solid

$$\partial_t^2 \mathcal{K}^{-1}_{ij}\phi_j = \underbrace{\lambda_{ijkl}\partial_j\partial_k\phi_l}_{\text{elastic}} + \underbrace{\eta_{ijkl}\partial_t\partial_j\partial_k\phi_l}_{\text{viscous}}$$

In an isotropic solid:

$$\eta_{ijkl} = \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\right) + \zeta\,\delta_{ij}\delta_{kl}$$

$$\lambda_{ijkl} = \underset{\text{shear}}{\mu}\left(\qquad\qquad\qquad\qquad\right) + \underset{\text{bulk}}{K}\,\delta_{ij}\delta_{kl}$$

$$\mathcal{K}^{-1}_{ij} = \rho\,\delta_{ij}$$ mass density

Quasinormal modes: $\phi \sim e^{ikx - i\omega t}$

longitudinal $\vec{\phi} \parallel \vec{k}$: (P wave) (sound)

$$\rho\omega^2\phi_x = \left(\frac{2d-2}{d}\mu + K\right)k^2\phi_x - i\omega k^2\left(\frac{2d-2}{d}\eta + \zeta\right)\phi_x$$

transverse: $\phi \perp k$: (S wave) (sound)

$$\rho\omega^2\phi_\perp = \mu k^2\phi_\perp - i\omega k^2\eta\,\phi_\perp$$

e.g. S wave: $\omega \approx \pm k\underbrace{\sqrt{\frac{\mu}{\rho}}}_{v_s} - i\frac{\eta}{\rho}k^2 + \cdots$

Unlike ordinary fluid... Sound in transverse direction.

Note: $v_p = \sqrt{\dfrac{4/3\,\mu + K}{\rho}}$ $(d=3)$

$$\frac{v_p}{v_s} \geq \sqrt{\frac{4}{3}}$$ since $\mu, K \geq 0$.

e.g. earthquake physics: P waves hit first
S waves second

Clear from EFT: solids have viscosity.

$$\frac{\eta}{\rho} \sim 1\ \frac{m^2}{s} \longleftrightarrow \frac{\eta}{\rho} \sim 10^{-6}\ \frac{m^2}{s}$$ vs water

kinematic viscosity