PHYS 7810
Hydrodynamics
Spring 2024

Lecture 28
Fracton hydrodynamics

April 30

Recall: simple theory of diffusion (single cons. charge) (cf lec 6)

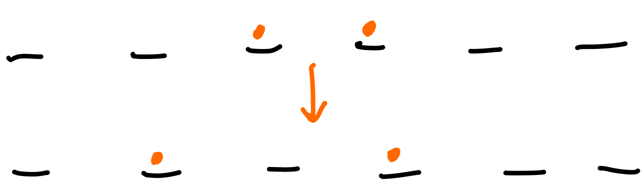
If $\frac{d}{dt}\int dx\, \rho = 0$ then

↳ Noether's Thm: $\pi \to \pi + 1$ shift sym

$$\mathcal{L}_{MSR} = \pi\, \partial_t \rho + i\sigma\, \partial_x \pi\, \partial_x(\pi - i\mu) + \cdots$$

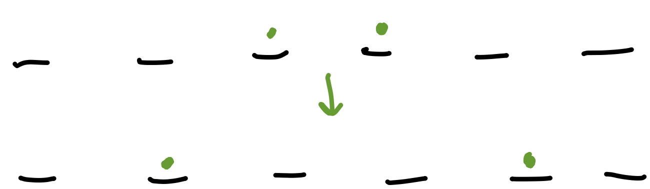
Let's add more interesting "space-dependent" cons. laws...

just charge



charge cons: $\sum q_x$ ($q_x = 0$ or 1)

charge + "dipole"



dipole: $\sum x\, q_x$

"Fracton" hydro $\Rightarrow$ microscopic model has immobile (reduced mobility) single particles.

Propose: in hydro limit we need:

charge cons $\frac{d}{dt}\int dx\, \rho(x) = 0$

$+$

dipole cons $\frac{d}{dt}\int dx\, x\,\rho(x) = 0$

Goal: $\mathcal{L}_{MSR} = \pi \partial_t \rho + H(\pi, \rho)$ ← what invariant building blocks?

Abstract: $\frac{d}{dt}\underbrace{\int dx\, \rho(x) f_\alpha(x)}_{Q_\alpha} = 0 \ldots$ for some set $f_\alpha \ldots$

Noether Thm: $\pi \to \pi + \frac{\delta Q_\alpha}{\delta \rho} = \pi + f_\alpha(x)$

Suppose found diff operator $\mathcal{D}$ s.t. $\mathcal{D} f_\alpha = 0$ for all α.

Then: $\mathcal{L} = \pi\partial_t\rho + i\sigma\,\underbrace{(\mathcal{D}\pi)}_{\text{obeys Noether}}\,\underbrace{\mathcal{D}(\pi - i\mu)}_{\text{time-reversal!}} + \cdots$

Assume: $\Phi = \int dx\, \frac{\rho^2}{2\chi} \ldots \quad \mu = \rho/\chi$. Then

hydro EOM: $\partial_t \rho = -\frac{\sigma}{\chi}\mathcal{D}^T\mathcal{D}\rho$

Example 1: dipole + charge

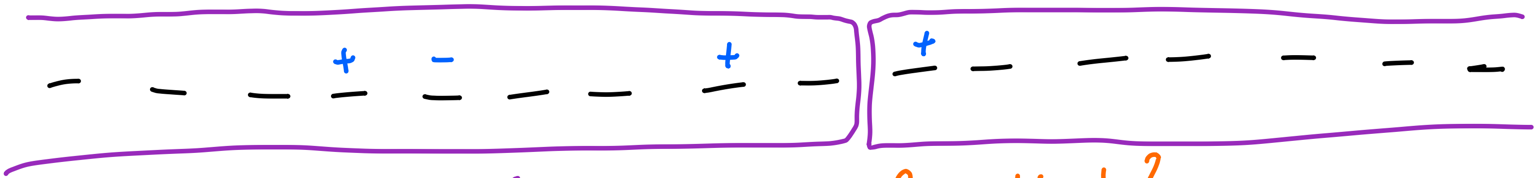
$f = x$ (dipole), $f = 1$ (charge)

Simplest: $\mathcal{D} = \partial_x^2 \rightsquigarrow \partial_x^T = -\partial_x \qquad (\partial_x\partial_x)^T = \partial_x^T\partial_x^T = \partial_x^2$

so: $\partial_t \rho = -\frac{\sigma}{\chi}\boxed{\partial_x^4}\rho \rightsquigarrow$ subdiffusion

easily testable signature of dipole conservation.

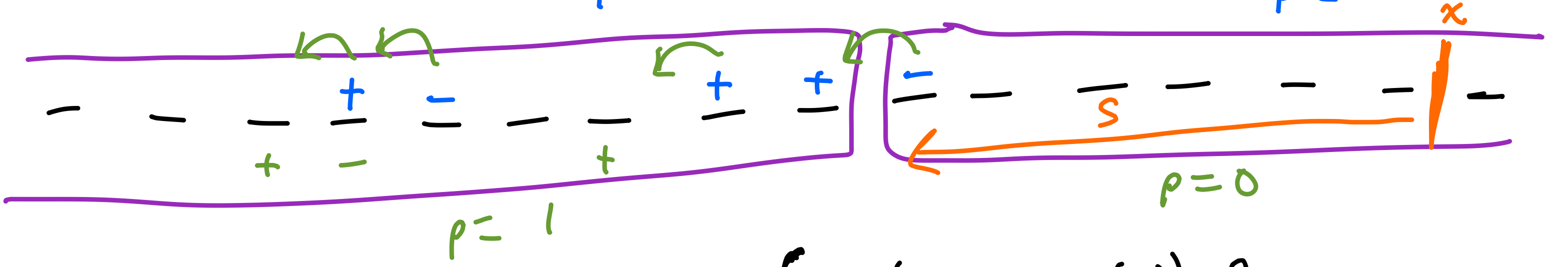
Why don't we worry about bound dipoles?



coarse grain into hydro boxes: & dipole?

$$\rho(x) \sim \sum_{j \in \text{box } x} q_j \qquad s(x) \sim \sum_{j \in \text{box } x} (j-x)\, q_j$$

$\rho = 2$ $\qquad$ $\rho = -1$



Overall dipole moment:
$$\left.\begin{aligned} D &= \int dx\, \left(x\rho(x) + s(x)\right) \\ Q &= \int dx\, \rho(x) \end{aligned}\right\} \text{ conserved.}$$

If $\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_s \partial_t s + \cdots$?

Using Noether: invariant under $\begin{pmatrix}\pi_\rho \\ \pi_s\end{pmatrix} \to \begin{pmatrix}\pi_\rho \\ \pi_s\end{pmatrix} + \underbrace{\begin{pmatrix}1 \\ 0\end{pmatrix}}_{\text{charge}} + \underbrace{\begin{pmatrix}x \\ 1\end{pmatrix}}_{\text{dipole}}$

Invariant motifs? $\quad \partial_x^2 \pi_\rho,\ \partial_x \pi_s,\ \partial_x \pi_\rho - \pi_s$

Assume: $\mu_\rho \simeq \dfrac{\rho}{\chi}$ and $\mu_s = \dfrac{s}{\chi'}$:

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_s \partial_t s + i\gamma_1 (\partial_x \pi_\rho - \pi_s)\left(\partial_x(\pi_\rho - i\mu_\rho) - (\pi_s - i\mu_s)\right) + i\gamma_2 \partial_x \pi_s\, \partial_x(\pi_s - i\mu_s) + \cdots$$

$\downarrow$

$$\partial_t \rho = \gamma_1 \partial_x \left(\frac{\partial_x \rho}{\chi} - \frac{s}{\chi'}\right) + \cdots$$

$$\cancel{\partial_t s} = -\underline{\underline{\gamma_1}} \left(\frac{s}{\chi'} - \frac{\partial_x \rho}{\chi}\right) + \gamma_2 \partial_x^2 \frac{s}{\chi'} + \cdots$$

If $\underbrace{\partial_t \ll \gamma_1}_{\text{hydro limit!}}$: $\quad \gamma_1 \left(\dfrac{s}{\chi'} - \dfrac{\partial_x \rho}{\chi}\right) \simeq \gamma_2 \partial_x^2 \dfrac{s}{\chi'}$

Same subdiffusion!

$$\to \partial_t \rho = \partial_x \left(-\gamma_2 \partial_x^2 \frac{s}{\chi'}\right) \approx -\gamma_2 \partial_x \partial_x^2 \partial_x \frac{\rho}{\chi} = -\frac{\gamma_2}{\chi} \partial_x^4 \rho$$

angular momentum (cf lec 9) ~ "antisymmetric dipole of momentum": $L_{ij} = x_i g_j - x_j g_i$

Where does dipole cons. show up in Nature?

- slow charge subdiffusion in tilted lattice models (Bloch oscillations)
- surface growth in crystals

Example 2: multipole conservation:

Suppose $Q_m = \int dx\, x^m \rho(x)$ conserved. [for $m = 0, 1, \dots, n$]

Then take: $\underline{\mathcal{D} = \partial_x^{n+1}}$ $[1 + x + \dots x^n] = 0$

hydro equation: $\partial_t \rho + \frac{\sigma}{\chi}(-1)^{n+1} \partial_x^{2n+2} \rho = 0.$

Does it make sense to have Q_0 & Q_2 cons. but not Q_1?

Need $\mathcal{D}(a + bx^2) = 0$ but $\mathcal{D} x \neq 0$.

What 2nd order linear ODE solved by $a + bx^2$?

$$\mathcal{D} = \left[\left(\partial_x - \tfrac{1}{x}\right)\overset{2x}{\partial_x}\right]$$

→ not spatially homogeneous!

To have spatial homogeneity, need:

$$\mathcal{D} = a_m \partial_x^m + a_{m-1}\partial_x^{m-1} + \dots + a_0$$

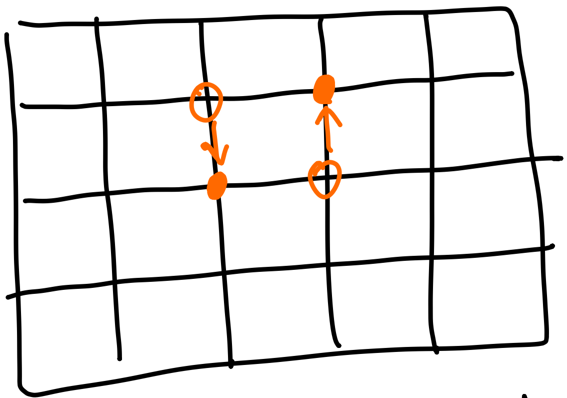
General solution: $\{e^{\lambda x} \cdot \mathrm{poly}(x)\}$. Therefore all $\underline{\underline{1d}}$ spatially homogeneous fracton hydro models (not sick at ∞):

- multipole-conserving models: $\int \rho, \int x\rho, \dots \int x^m \rho$

(and) - modulated symmetry: $\int \cos(k_0 x)\rho$ & $\int \sin(k_0 x)\rho$ (HW 2)

Example 3: subsystem symmetry $(d>1)$

↳ charge conserved on submanifolds



charge conserved on each row & column.

In hydro limit: $0 = \frac{d}{dt}\int_{x=a} \rho(x,y)\,dy = \frac{d}{dt}\int_{y=b} \rho(x,y)\,dx$

$f=\delta(x-a),\ g=0.$

Rewrite: $0 = \frac{d}{dt}\int dx\,dy\ \rho(x,y)\,\underbrace{\left[f(x) + g(y)\right]}$

For what $\mathcal{D}$ is $f(x)+g(y)$ general solution to $\mathcal{D}F(x,y)=0$.

Take: $\mathcal{D} = \partial_x \partial_y$. Then: $\partial_t \rho = -\frac{\sigma}{\chi}\partial_x^2\partial_y^2 \rho$

4th order subdiff. but not rotation invariant.

This equation also has UV/IR mixing.

Suppose keep only slow DOF w/ $\underbrace{|\omega| \le \omega_0 \to 0}$.

$$\omega = -i\frac{\sigma}{\chi}k_x^2 k_y^2$$

$$\omega_0 \ge \frac{\sigma}{\chi}k_x^2 k_y^2$$

Keep modes at any k_x if $k_y \le \sqrt{\frac{\chi\omega_0}{\sigma}}\,\frac{1}{k_x}$

$\sim$ lattice scale?

This EFT works (numerical prediction) despite UV/IR mixing.