PHYS 7810
Hydrodynamics
Spring 2024

Lecture 29
Higher-form symmetries

May 2

Fracton hydro (in higher dimensions):

$$\mathcal{L}_{MSR} = \pi \partial_t \rho + i \kappa \underbrace{\mathcal{D}\pi}\, \mathcal{D}(\pi - i\mu)$$

choose differential operator $\mathcal{D}$ so that $\mathcal{D}f = 0$

where $\frac{d}{dt}\int d^d x \;\rho \cdot f(x) = 0$.

... up to add'l indices on $\mathcal{D}$

e.g. for dipole conserving: $\frac{d}{dt}\int d^d x \;\rho \overbrace{\left[a + b_i x_i\right]}^{f} = 0$.

↳ Step 1: demand conserved quantities.

Step 2: Find $\mathcal{D}$ annihilates all f:

Take $\mathcal{D} = \partial_i \partial_j$ and write:

$$\mathcal{L} = \pi \partial_t \rho + \kappa\, i\, \partial_i \partial_j \pi\, \partial_i \partial_j (\pi - i\mu) \qquad \text{where } \Phi = \int d^d x \,\frac{\rho^2}{2\chi}$$

← + other index contractions

$$\hookrightarrow \quad \partial_t \rho + \frac{\kappa}{\chi} \partial_i \partial_j \partial_i \partial_j \rho = 0$$

subdiffusive: $\rho \sim e^{ikx - i\omega t}$

$$\omega = -i \frac{\kappa}{\chi} k^4$$

An alternate philosophy:

Step 1: $\mathcal{D} = \partial_i \partial_j$

Step 2: what f obey $\partial_i \partial_j f = 0$ for all i & j?

$\hookrightarrow f = a + b_i x_i$

Most general hydro EFT:

$$\mathcal{L}_{MSR} = \pi \partial_t \rho + \mathcal{D}_{ij}\pi \cdot \sigma J_{ij}$$

generalization of current

$\hookrightarrow$ EOM: $\partial_t \rho + \mathcal{D}^T \sigma J = 0$

dipole: $\underline{\partial_t \rho + \partial_i \partial_j J_{ij} = 0}$ modified Ward identity

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Example 1:

Try $\mathcal{D} = \nabla^2 = \partial_i \partial_i$

Conserved quantities: $\mathcal{D} f = 0 = \nabla^2 f \Rightarrow f$ is any harmonic function

In $d=2$, already ∞:

$$f \in \{1, x, y, xy, x^2-y^2, x^3 - 3xy^2, \ldots\}$$

Hydro EOM identical to dipole-conserving theory...

$$0 = \partial_t \rho + \frac{\kappa}{\chi}(\nabla^2)^2 \rho$$

Example 2: go back to old perspective?

$$f \in \{1, x, y, x^2-y^2, xy\}$$

NB: (cf lec 28) $\hookrightarrow$ any f in this list has $\partial_x f$ & $\partial_y f$ also conserved!

Guess... $\mathcal{D} = \nabla^2$ ... but extra conservation laws are unwanted?

Add irrelevant perturbations to $\mathcal{L}$: dangerously irrel.

$$\mathcal{L} = \pi \partial_t \rho + i\kappa \nabla^2 \pi \nabla^2 (\pi - i\mu) + i\kappa' \partial_i \partial_j \partial_k \pi \, \partial_i \partial_j \partial_k (\pi - i\mu)$$

harmonic $\qquad\qquad$ $f \in \{1, x, \ldots, x^2, y^2, xy\}$

intersection is the desired set of conserved q's.

Irrelevant perturbation distinguishes from all-harmonic-conserved.

What happens if ρ has "indices".

Today: ρ_i as a vector under spatial rotation.

If $\frac{d}{dt}\int d^d x \, \rho_i f_i = 0 \ldots \qquad \pi_i \to \pi_i + f_i$

w/ $\mathcal{L}_{MSR} = \pi_i \partial_t \rho_i + (\mathcal{D}\pi)_i \cdot [\cdots]$, need $(\mathcal{D} f) = 0$.

Many more possibilities!

Now work to $d=3$. Back to "alternate" approach:

$$\mathcal{D}\pi \, \mathcal{D}(\pi - i\mu) \rightarrow \partial_i \pi_j \cdot P_{ij,kl} \, \partial_k (\pi_l - i\mu_l)$$

decompose "3×3 matrix $\partial_i \pi_j$" into $SO(3)$ irreps:

"spin 0": $P_{ijkl} = \delta_{ij}\delta_{kl}$ trace

$\partial_i \pi_i \, \partial_j (\pi_j - i\mu_j)$

"spin 1": $P_{ijkl} = \varepsilon_{ijn}\varepsilon_{kln}$ antisym. matrices

"spin 2": $P_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}$ (traceless symmetric)

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{more interesting.}}$

Spin-1 condition: $\mathcal{D}f = 0 \rightsquigarrow \underbrace{\varepsilon_{ijm}\partial_i f_j}_{\nabla\times\vec{f}} = 0.$

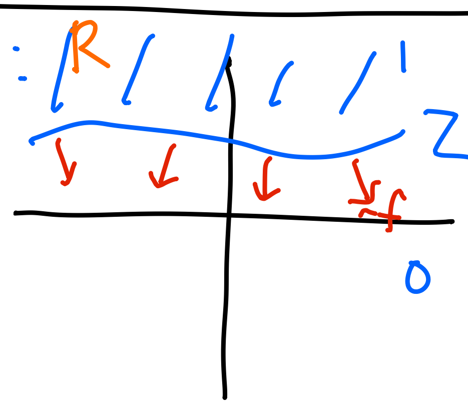
Solved by: $\vec{f} = \nabla\Lambda$

Spin-0: $\mathcal{D}f = 0 \rightsquigarrow \partial_i f_i = 0$ or $\nabla\cdot\vec{f} = 0$

or $\vec{f} = \nabla\times\vec{A}$

Interpretation? For spin-1, helpful to take:

$$\Lambda = \Theta(z - Z(x,y))$$



$$\vec{f} = (-\partial_x Z, -\partial_y Z, 1)\cdot\delta(z-Z)$$

Conserved quantity:

$$\int_R d^3x\ \rho_i\partial_i\Lambda = \underbrace{\oint_{\partial R} d^2x\ n_i\rho_i}$$

= flux of ρ through boundary ∂R (closed 2d surface)

For spin-0: $$A_k(x) = \frac{1}{4\pi}\oint_{\text{curve } C} ds\ \varepsilon_{ijk}\overset{\text{runs along } C}{\frac{\partial y_j(s)}{\partial s}}\frac{x_i - y_i(s)}{|x-y(s)|^3}$$

Biot-Savart: $\nabla\times\vec{A} = \vec{f} \rightsquigarrow$ unit "current" around C.

Cons. q: $\frac{d}{dt}\oint_C dl_i\,\rho_i = 0$ for all curves C.

Nice geometric formulation of problem using differential forms.

back to usual charge conserv:

$$0 = \partial_t\rho + \partial_i J_i = \partial_\mu J^\mu$$

Define 1-form $J = J_\mu dx^\mu \longrightarrow 0 = d*J$

What if J is a p-form?

$$*dx_1 = dx_2 \wedge dx_3$$

$$J = \underbrace{J_{\mu_1 \cdots \mu_p}}_{\text{antisymmetrized}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

Then $*d*J = 0 \rightsquigarrow \underline{\partial_\nu J^{[\nu\mu_1 \cdots \mu_p]} = 0.}$

Take: $\mu_1 \cdots \mu_{p-1} = \underbrace{i_1 \cdots i_{p-1}}_{\text{Spatial!}}$; $\partial_t \underbrace{J^{t i_1 \cdots i_{p-1}}}_{\rho^{[i_1 \cdots i_{p-1}]}} + \partial_j J^{[j i_1 \cdots i_{p-1}]} = 0$

all indices distinct

Take $\mu_1 \cdots \mu_{p-1} = i_1 \cdots i_{p-2} t$: $\partial_j J^{[j i_1 \cdots i_{p-2} t]} = 0$

$$\partial_j \rho^{[j i_1 \cdots i_{p-2}]} = 0$$

a constraint on density!

This theory has higher-form symmetry [$(p-1)$-form sym.]

"Spin-1 theory" $\longrightarrow$ 1-form symmetry.

$$0 = \partial_t \rho_i + \epsilon_{ijk} \partial_j J_k = \partial_t \rho_i + \partial_j \underbrace{J_{[ij]}}_{\text{like 1-form symmetry...}}$$

$$\frac{d}{dt} \int d^3x \, \rho_i \partial_i \Delta = 0 \quad \text{or} \quad -\frac{d}{dt} \int d^3x \, \Delta(x) \partial_i \rho_i = 0.$$

Reasonable to take $\underline{\partial_i \rho_i = 0}$ [or $\partial_i \rho_i$ t-indep.]

only MSR!

Spin-0 theory $\rightarrow$ 2-form symmetry

Idea: higher-form symmetries = conservation laws on subdimensional manifolds.

For spin-1 theory, quasinormal modes: $\rho_i \sim e^{ikx - i\omega t}$

ρ_y & ρ_z: $\omega \sim -iDk^2$ $(\rho \perp k)$

ρ_x : $\omega = 0$ $(\rho \parallel k)$ $(k_i \rho_i = 0)$

Physical realization of higher-form symmetry:

minimal model of magnetohydrodynamics.

$\rightarrow$ magnetic fields in metals

Since $\nabla \cdot \vec{B} = 0$ $[\partial_i B_i = 0]$ so B_i might be "1-form charge?"

Faraday's Law: $\partial_t B_i + \underbrace{\varepsilon_{ijk} \partial_j E_k}= 0$

E is "higher-form current"

In a conductor, $J_i = \sigma E_i$

Ampere's Law: $\mu J_i = \mu \sigma E_i = \varepsilon_{ijk} \partial_j B_k$

Combine to get theory of magnetic diffusion:

$$\partial_t B_i = \mu \sigma [\partial_j \partial_j B_i - \partial_i \cancel{\partial_j B_j}]$$

This reproduces MSR diffusive theory for 1-form:

$$\mathcal{L}_{MSR} = \pi_i \partial_t \rho_i + i \kappa \varepsilon_{ijm} \varepsilon_{klm} \partial_i \pi_j \partial_k (\pi_l - i \mu_l)$$