PHYS 7810
Hydrodynamics
Spring 2024

Lecture 3
Time-reversal symmetry

January 23

Recap: stochastic differential equation:

$$\frac{dx}{dt} = a(x,t) + b(x,t)\underbrace{\xi(t)}$$

Gaussian white noise:
$\langle \xi(t) \rangle = 0$
$\langle \xi(t)\xi(t') \rangle = \delta(t-t')$.

$\downarrow$ Ito prescription

Fokker-Planck equation:

$$\partial_t P = -\partial_x(aP) + \partial_x^2\left(\frac{1}{2}b^2 P\right)$$

$P(\tilde{x},t)$ = probability of finding $x(t) = \tilde{x}$.

Example: overdamped oscillator.

$$\frac{dx}{dt} = -\gamma x + \sqrt{2D}\,\xi(t) \qquad \gamma, D>0 \text{ const.}$$

FPE $\downarrow$

$$\partial_t P = \partial_x(\gamma x P) + D\partial_x^2 P$$

Suppose $P(x,0) = \delta(x-y)$. What is $P(x,t)$?

Fourier transform to $\hat{p}(k,t)$:

$$\partial_t \hat{p} = -\gamma k\, \partial_k \hat{p} - D k^2 \hat{p}$$

Solved using method of characteristics:

$$\frac{dk}{dt} = +\gamma k \qquad\qquad \frac{d\hat{p}}{dt} = -D k^2 \hat{p}$$

$$k(t) = k(0) e^{\gamma t} \qquad\qquad \frac{d\hat{p}}{\hat{p}} = -D k(0)^2 e^{2\gamma t}\, dt$$

$$\log \frac{\hat{p}(k,t)}{\hat{p}(k,0)} = -D k(0)^2 \frac{e^{2\gamma t} - 1}{2\gamma}$$

Combine:

$$\hat{p}(k,t) = \hat{p}(k e^{-\gamma t}, 0) \exp\left[-D k^2 \frac{1 - e^{-2\gamma t}}{2\gamma}\right]. \qquad \hat{p}(k,0) = e^{iky}$$

$$\hookrightarrow P(x,t) \sim \exp\left[-\frac{\gamma}{2D} \frac{(x - y e^{-\gamma t})^2}{2(1 - e^{-2\gamma t})}\right].$$

As $t \to \infty$:

$$P_{ss}(x) = P(x, t \to \infty) \sim \exp\left[-\frac{\gamma}{2D} x^2\right].$$

$\beta = \frac{1}{T}$

Reminds us of canonical ensemble: $P_{ss}(x) \sim e^{-\beta H(x)}$

$H = \frac{1}{2} k x^2$.

$$\frac{\gamma}{2D} x^2 = \frac{1}{T} \cdot \frac{1}{2} k x^2, \quad \text{or:}$$

$$\frac{k}{T} = \frac{\gamma}{D}$$: fluctuation–dissipation theorem.

In general: "any" stochastic system will have steady state:

$$\underbrace{\partial_t P_{ss}(x)}_{} = -\partial_x (a P_{ss}) + \cdots = 0.$$

$\hookrightarrow P_{ss} = e^{-\Phi}$ $\qquad$ $\Phi \sim$ "βH" $\sim$ free energy

Note: FPE are linear: $P(x,t)$ solution $\to$ $\lambda P(x,t)$.
(overall normalization not important)

Example: re-do overdamped oscillator.

Start by assuming steady state $P_{ss} = e^{-\Phi} = e^{-\beta H} = e^{-\frac{1}{2}\frac{k}{T}x^2}$.

The most general FPE: $\partial_t P = -\hat{W} P \ldots$

Most general $\hat{W}$?

① conserve probability:

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, P = 0 = -\int_{-\infty}^{\infty} dx\, \hat{W} P \ldots$$

$$\int_{-\infty}^{\infty} dx\, \partial_x(\hat{Y}P) = \hat{Y}P\Big|_{-\infty}^{+\infty} = 0.$$

$$\hat{W}P = \partial_x(\hat{Y}P)$$

if $P(\pm\infty, t) = 0$, then $\hat{W} = \vec{\partial}_x \hat{Y}$

② stationarity: $\hat{W} e^{-\Phi} = 0$.

Claim: $\hat{W} = \hat{Z}(\vec{\partial}_x + \partial_x \Phi)$

$$(\vec{\partial}_x + \partial_x\Phi)e^{-\Phi} = -(e^{-\Phi})\partial_x\Phi + \partial_x\Phi\, e^{-\Phi} = 0.$$

How do I obey both?

$$\hat{W} = -\vec{\partial}_x \underset{\text{arbitrary}}{\hat{Q}} (\vec{\partial}_x + \mu) \quad \text{where } \mu = \partial_x\Phi$$

In this example: $\mu = \frac{k}{T}x$.

Simplest possible choice: $\hat{Q}P = DP$, D: diffusion const. $(D>0)$ noise variance >0.

$$-\hat{W}P = \partial_t P = D\partial_x^2 P + D\partial_x\left(\frac{k}{T}xP\right)$$

$$\hookrightarrow + \partial_x(\gamma x P)$$

γ obeys FDT!

$$\gamma = D\cdot\frac{k}{T}$$

Latter approach: philosophy of effective theory.

Usually take "$\hat{Q}$" to be const...

- multiplicative noise: $\hat{Q} P = Q(x) P$ ($\neq$ const.)
- non-Gaussian noise: $\hat{Q}$ have ∂_x (FPE has $\geq 3^{rd}$ derivatives) $\hookrightarrow \infty$ order
- Non-Markovian noise: FPE's w/ $\cancel{\partial_t^2} + \dots$ (not in class)

$$\partial_t P = -\int_0 ds\, \hat{W}(s) P(t-s)$$

Generalization to n degrees of freedom:

① given stationary state $e^{-\Phi}$ (usually $\Phi = \beta H$)

$\downarrow$

② most general FPE: $\partial_t P = -\hat{W} P$

$$\hat{W} = -\partial_i \hat{Q}_{ij} (\partial_j + \mu_j) \quad \text{where } \mu_j = \partial_j \Phi.$$

$(i = 1, \dots, n)$

The operator $\hat{Q}_{ij}$ has to be positive semidefinite (noise variance not negative)

$$\rightarrow P(x, t\to\infty) = P_{ss}(x)$$

Note: above $\hat{W} = -\vec{\partial}_i \hat{Q}_{ij} (\vec{\partial}_j + \partial_j \Phi)$ not in Itô prescription!

$$-\hat{W}_{Ito} P = -\partial_i (a_i P) + \partial_i \partial_j (Q_{ij} P) + \dots$$

So far: FPE seems to have an arrow of time:

$$P(x, \underbrace{t\to\infty}_{(\text{NOT } -\infty)}) = P_{SS}(x)$$

Claim: this is a consequence of our ~~initial conditions~~! knowledge of $P(x,0)$.

Dissipative systems can have time-reversal symmetry... only w/ stochastic noise!

Recall: $P(x,0) = \delta(x-y)$. Then

$P(x,t) = \langle x| e^{-\hat{W}t} |y\rangle$ = prob. to go from y to x in Δ(time) t.

If we watch "movie" of dynamics from y to x... Can we tell arrow of time?

$(x,0) \to (y,t)$:

$$\langle y| e^{-\hat{W}t} |x\rangle = \langle x| e^{-\hat{W}^T t} | y\rangle$$

are they equal?

$A^T B^T = (BA)^T$

$\partial^T = -\partial$

No. In general not same because:

$$\hat{W} = -\partial_i \underbrace{Q_{ij}}_{\text{Gaussian}} (\partial_j + \mu_j) \qquad \hat{W}^T = (\partial_j - \mu_j) Q_{ij} \partial_i$$

Bad because:

W^T is...

- not stochastic process (not $\partial_i(\cdots)$)

[- steady state messed up $W^T e^{-\Phi} \neq 0$].

Let's try to fix! Define time-reversed

$$\hat{W}^{rev} = e^{-\Phi} \hat{W}^T e^{\Phi}$$

(similarity transformation, multiplying by $P_{ss}^{\pm 1}$)

Claim: $\hat{W}^{rev}$:
- stochastic!
- steady state unchanged!

$$\hat{W}^{rev} = e^{-\Phi} \hat{W}^T e^{\Phi} = e^{-\Phi} (\partial_j - \mu_j) Q_{ij} \partial_i e^{+\Phi}$$

$$\underbrace{e^{-\Phi}(\partial_j - \mu_j) e^{\Phi}}_{\partial_j} \; \underbrace{e^{-\Phi} Q_{ij} e^{\Phi}}_{Q_{ij}} \; \underbrace{e^{-\Phi} \partial_i e^{\Phi}}_{\partial_i + \mu_i}$$

i.e. $\hat{W}^{rev} = -\partial_j Q_{ij} (\partial_i + \mu_i)$

$$= -\partial_i Q^T_{ij} (\partial_j + \mu_j)$$

$\hat{W}^{rev}$ generates the time-reversed process.

Why? $\langle x | e^{-\hat{W} t} | y \rangle = \langle y | e^{-\hat{W}^T t} | x \rangle$

$$= \langle y | e^{\Phi} \underbrace{e^{-\Phi} e^{-\hat{W}^T t} e^{\Phi}}_{} e^{-\Phi} | x \rangle$$

$$e^{+\Phi(y)} \langle y | e^{-\hat{W}^{rev} t} | x \rangle e^{-\Phi(x)}$$

Define: stochastic process has T-symmetry if

$$\hat{W}^{rev} = \hat{W}.$$

Then obey detailed balance:

$$\langle x | e^{-\hat{W} t} | y \rangle e^{-\Phi(y)} = \langle y | e^{-\hat{W} t} | x \rangle e^{-\Phi(x)}$$

Notice that if T-symmetry:

$$\hat{W} = \hat{W}^{rev} \rightsquigarrow Q_{ij} = Q_{ji} \quad \text{(symmetric } Q\text{)}$$

$$\hat{W} = -\partial_i Q_{ij} \left(\partial_j + \mu_j \right)$$

fluctuation (∂_j) dissipation (μ_j)

Puzzle about of arrow of time has resolution:

$t=0$ relaxes to "eq." $t=\infty$

no "un-thermalize"

$\underline{e^{-\Phi} \text{ small}}$ $\underline{e^{-\Phi} \text{ large}}$