PHYS 7810
Hydrodynamics
Spring 2024

Lecture 4
MSR Lagrangian

January 27

Recap: two perspectives on stochastic equations:

① Itō prescription:

$$\dot{x}_i = a_i + b_{i\alpha}\xi_\alpha(t)$$

Gaussian white noise: $\langle \xi_\alpha(t)\rangle = 0$, $\langle \xi_\alpha(t)\xi_\beta(t')\rangle = \delta_{\alpha\beta}\,\delta(t-t')$

Fokker–Planck equations

$$\partial_t P = -\hat{W}P = -\partial_i(a_i P) + \frac{1}{2}\partial_i\partial_j(b_{i\alpha}b_{j\alpha}P)$$

② postulate stationary state $P_{ss} = e^{-\Phi}$, assert FPE:

$$\partial_t P = -\vec{\partial}_i\left(Q_{ij}^{[x]}(\vec{\partial}_j + \mu_j)P\right) \quad \text{where } \mu_j = \partial_j\Phi$$

assume Gaussian noise

Time-reversal: $Q_{ij}^{rev} = Q_{ji}$ $\longrightarrow$ If $Q = Q^T$, time-reversal sym.

Detailed balance:

$$\langle x|e^{-\hat{W}t}|y\rangle e^{-\Phi(y)} = \langle y|e^{-\hat{W}t}|x\rangle e^{-\Phi(x)}$$

Fluctuation-dissipation theorem: (F) (D) if we have noise (F)... must be dissipative (D) as well.

Goal: build $\underbrace{\text{effective theory}}_{\text{emergent / coarse-grained description: slow DOF}}$ of $\underbrace{\text{dissipative + noisy}}_{\text{for t-ind./statistical steady state}}$ systems

E.g. start w/ classical Hamiltonian...

$$H(x_1,\ldots,x_N,p_1,\ldots,p_N) = \sum_{\alpha=1}^{N} \frac{p_\alpha^2}{2m} + \sum_{\alpha<\beta} V(|x_\alpha - x_\beta|)$$

① Spot slow DOF directly. $(\bar{x},\bar{p})$: center of mass position, total momentum

② ensure appropriate symmetries...

Ⓐ Time-reversal symmetry.

$$\left.\begin{aligned} x_\alpha &\to x_\alpha \\ p_\alpha &\to -p_\alpha \\ t &\to -t \end{aligned}\right\} \quad \begin{aligned} \bar{x} &\to \bar{x} \\ \bar{p} &\to -\bar{p} \\ t &\to -t \end{aligned}$$

"Integrated out" many microscopic DOF: $(x_1,\ldots,p_N) \to (\bar{x},\bar{p})$

↓

effective theory has dissipation & noise

Our FPE for $\bar{x}$ & $\bar{p}$ must have T: $\partial_t P = -\partial_i Q_{ij}(\partial_j + \mu_j)P \ldots$

What Q_{ij} possible? If $\partial_i^T = \left(\frac{\partial}{\partial\bar{x}}, -\frac{\partial}{\partial\bar{p}}\right)$:

$$\hat{W}^{\text{rev}} = -\partial_i^T Q_{ji}(\partial_j^T + \mu_j^T) = \hat{W}$$

$$\to (\partial_{\bar{x}} \quad \partial_{\bar{p}}) \underbrace{\begin{pmatrix} (+1)^2 Q_{xx} & (+1)(-1)Q_{px} \\ (+1)(-1)Q_{xp} & (-1)^2 Q_{pp} \end{pmatrix}}_{= Q_{ij}} \begin{pmatrix} \partial_{\bar{x}} + \mu_{\bar{x}} \\ \partial_{\bar{p}} + \mu_{\bar{p}} \end{pmatrix}$$

(symmetric)

dissipative coefficients:

$Q_{xx}, Q_{pp} \geq 0$ (noise variances)

(antisymmetric)

$Q_{px} = -Q_{xp}$

dissipationless

Ⓑ strong symmetry (conservation law): $\frac{dF}{dt} = 0$ (on each trajectory)
(weak symmetry) (on avg)

↳ e.g. momentum

e.g. $\langle \frac{dx}{dt} \rangle = 0$ for random walk.

If F is conserved on every trajectory:

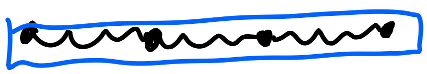
$$P_{ss} = e^{-\Phi - \lambda F} \quad \text{for any } \lambda.$$

$$\partial_t \left(P e^{-\lambda F} \right) = \overbrace{e^{-\lambda F}}^{F \text{ cons.}} \partial_t P = e^{-\lambda F} \left(-\partial_i Q_{ij} (\partial_j + \mu_j) P \right)$$

$$= -\left(e^{-\lambda F} \partial_i e^{\lambda F} \right) \left(e^{-\lambda F} Q_{ij} e^{\lambda F} \right) \left(e^{-\lambda F} (\partial_j + \mu_j) e^{\lambda F} \right) \left(e^{-\lambda F} P \right)$$

$$\underbrace{-(\partial_i + \lambda \partial_i F) \cdot Q_{ij} \cdot (\partial_j + \lambda \partial_j F + \mu_j)}_{= -\partial_i Q_{ij} (\partial_j + \mu_j)} \left(e^{-\lambda F} P \right)$$

Or: FPE invariant under $\partial_i \to \partial_i + (\partial_i F)$

Example:  $(\bar{x}, \bar{p})$

has both time-reversal & momentum conservation.

Ⓐ time-reversal: $(\bar{x} \to x, \bar{p} \to p)$

$$\partial_t P = \partial_x Q_{xx} (\partial_x + \mu_x) P + \partial_p Q_{pp} (\partial_p + \mu_p) P$$
$$\underline{+ \partial_x Q_{xp} (\partial_p + \mu_p) P - \partial_p Q_{xp} (\partial_x + \mu_x) P}$$

Interested in <u>thermal</u> steady state: $\Phi = \beta H$

Dissipationless terms! Let's take $Q_{xp} = \text{const.}$

$$\partial_t P = \cdots \; Q_{xp} \beta \left[\partial_x \left(\frac{\partial H}{\partial p} P \right) - \partial_p \left(\frac{\partial H}{\partial x} P \right) \right]$$

$$\frac{\partial H}{\partial p} \frac{\partial P}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial P}{\partial p} = \{ P, H \}$$

Dissipation only are equivalent to:

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x} \qquad \text{if} \quad Q_{xp}\beta = -1$$

Hamilton's equations <u>are</u> dissipationless equation!

Ⓑ Momentum conservation (strong symmetry!)

$$\partial_x \to \partial_x + \cancel{\frac{\partial p}{\partial x}} \qquad \partial_p \to \partial_p + \frac{\partial p}{\partial p} = \partial_p + 1$$

Problematic terms:

$$(\partial_p + 1) Q_{pp} \left(\partial_p + 1 + \frac{\partial(\beta H)}{\partial p}\right) P + (\partial_p + 1)\frac{\partial H}{\partial x} P$$

↓ invariant if... ↓ invariant if...

$$Q_{pp} = 0 \qquad -\{p, H\} = \frac{\partial H}{\partial x} = 0 \quad \text{(Noether's Thm)}$$

The FPE formalism is exact ... but unwieldy.

Physical questions of interest:
- predicting noise / FDT
- <u>damping terms allowed?</u>

Goal: find "Lagrangian" effective theory?

Claim: natural Lagrangian for Itō SDEs.
($\approx$ simple symmetries)

More precisely:

$$\langle z | e^{-\hat{W}t} | y \rangle = \int_{\substack{x_i(t) = z \\ x_i(0) = y}} Dx_i \, D\pi_i \, e^{iS_{MSR}}$$

classical physics!

S_{MSR} action is Martin-Siggia-Rose (1973)

$$S_{MSR} = \int dt \, L_{MSR} .$$

Derivation – what's L_{MSR}? Suppose Itō SDE

$$\dot{x}_i = a_i + b_{i\alpha}\xi_\alpha(t)$$

Formally: $$\langle z|e^{-\hat{w}t}|y\rangle = \int_{BCs} \Big\langle \delta[\dot{x}_i - a_i - b_{i\alpha}\xi_\alpha] \Big\rangle_\xi Dx_i$$

subtle: Itō prescription needed to ensure $\int dz \, \langle z|e^{-\hat{w}t}|y\rangle = 1$.

$$\int \delta(x(t+\Delta t) - x(t) - [a+b\xi]\Delta t)\, dx(t+\Delta t) = 1 \quad \text{but} \quad \int \delta(\cdots)\, dx(t) \neq 1.$$

Perform average over ξ via path integral: $\langle \xi_\alpha\rangle = 0$.

$$\langle F\rangle_\xi = \int D\xi_\alpha \, F \, e^{-\int dt \, \xi_\alpha^2} \rightsquigarrow \langle \xi_\alpha(t)\xi_\beta(t')\rangle = \delta_{\alpha\beta}\delta(t-t')$$

Write $\delta[F_i] = \int D\pi_i \, e^{i\int dt \, \pi_i F_i}$. Combine:

$$\langle z|e^{-\hat{w}t}|y\rangle = \int_{BCs} Dx_i \int D\pi_i D\xi_\alpha \, e^{i\int dt\left[\pi_i(\dot{x}_i - a_i - b_{i\alpha}\xi_\alpha) + \frac{i}{2}\xi_\alpha^2\right]}$$

Perform path integral over ξ_α (quadratic):

Find EOM for ξ_α, plug in:

$$\frac{\delta S}{\delta \xi_\alpha} = 0 = i\xi_\alpha - \pi_i b_{i\alpha}$$

End result:

$$L_{MSR} = \pi_i\dot{x}_i - \pi_i a_i + \frac{i}{2}\pi_i\pi_j\tilde{Q}_{ij}$$

$$\tilde{Q}_{ij} = b_{i\alpha}b_{j\alpha}$$

$[\pi_i \sim -i\partial_i]$

Convert to FPE: $-\partial_i(a_i P)$ $\quad \frac{1}{2}\partial_i\partial_j(\tilde{Q}_{ij}P)$

π, x are now c-numbers... goal is to encode symmetries and build EFT using MSR Lagrangian!

How to build L_{MSR} as effective theorist?

Rule 0: $L_{MSR} = \pi_i \dot{x}_i - \mathcal{H}_{MSR}$ "Hamiltonian" (NOT H) $= \Phi/\beta$.

1: $\mathcal{H}_{MSR}$ encoding Itô SDE:

$\mathcal{H}_{MSR}(\pi_i = 0, x) = 0.$ and $\mathrm{Im}(\mathcal{H}_{MSR}) \leq 0.$

2A: Time-reversal symmetry:

if $x_i \to \pm x_i(-t)$ then $\pi_i \to \mp(\pi_i(-t) - i\mu_i(-t))$

$\mathcal{H}$ invariant under $\pi \to \mp(\pi - i\mu)$

2B: Strong symmetries:

$dF/dt = 0$: $\mathcal{H}$ invariant under $\pi_i \to \pi_i + \partial_i F(x)$

Example: (x, p) & T & p conserved.

$$L_{MSR} = \pi \dot{x} + \sigma \dot{p} + \underbrace{i(\pi \quad \sigma)\, Q \begin{pmatrix} \pi - i\mu_x \\ \sigma - i\mu_p \end{pmatrix}}_{-\mathcal{H}_{MSR}}$$

Q must be positive semidefinite.

Under time-reversal: $Q = \begin{pmatrix} Q_{xx} & Q_{xp} \\ -Q_{xp} & Q_{pp} \end{pmatrix}$

Same conclusions ...

Caveat: L_{MSR} gives us Itô SDEs. These are NOT

$$\partial_t P = \left[\partial_i(\partial_j + \mu_j)\, Q_{ij} P\right]$$

Discrepancy if $\partial_i Q_{ij} \neq 0$ (won't care usually).

From FPE to MSR?

$$\partial_t P = -\vec{\partial}_i\, Q_{ij}^{(x)}(\vec{\partial}_j + \mu_j)P$$

Itô: $$-\partial_i\left[\partial_j(Q_{ij} P) \underbrace{- (\partial_j Q_{ij})P + Q_{ij}\mu_j P}_{-\partial_i(a_i P)}\right]$$

$$\frac{1}{2}\partial_i\partial_j(2Q_{ij}P) \qquad a_i = Q_{ij}\mu_j - \partial_j Q_{ij}$$

MSR

!! : subtlety in regularization...

$$L_{MSR} = \pi_i \dot{x}_i + \pi_i\left(Q_{ij}\mu_j \boxed{-\partial_j Q_{ij}}\right) + i\,\pi_i \pi_j Q_{ij}$$