PHYS 7810
Hydrodynamics
Spring 2024

Lecture 5

Examples of MSR Lagrangians

January 30

Recap: Fokker-Planck equations, stationary state $e^{-\Phi}$:

$$\partial_t P = \partial_i \left[Q_{ij} (\partial_j + \mu_j) P \right] \qquad \text{where} \quad \mu_j = \partial_j \Phi$$

versus MSR Lagrangian: (Itō prescription)

$$L_{MSR} = \pi_i \dot{x}_i + i\pi_i Q_{ij} (\pi_j - i\mu_j) \; - \pi_i \partial_i Q_{ij}$$

ignore, usually for hydro

It's easier to work w/ MSR.

- $L_{MSR} = \pi_i \dot{x}_i - H_{MSR}$ where $H_{MSR}(\pi = 0) = 0$ (process is stochastic)

- noise variance non-negative: $\mathrm{Im}(H_{MSR}) \leq 0$
  imaginary part

- time-reversal: $x_i \to \pm x_i$ implies
  $\pi_i \to \mp(\underbrace{\pi_i - i\mu_i}_{\text{needed for detailed balance.}})$

time-reversed $\quad L_{MSR} = \mp(\pi_i - i\mu_i)(\mp \dot{x}_i) + i(\mp 1)(\pi_i - i\mu_i)\, Q_{ij}\, (\mp 1)\pi_j$

$$= \pi_i \dot{x}_i \underbrace{- i\frac{d\Phi}{dt}}_{\text{offset?}} + i\,\pi_i \underbrace{Q_{ji}}_{Q^T}(\pi_j - i\mu_j)$$

Q^T ← true (assume all x were T even (odd))

FPE: $\partial_t P = -\hat{W} P$

$$\langle z| e^{-\hat{W}t} |y\rangle = \int_{\substack{x(0)=y \\ x(t)=z}} Dx\, D\pi\, e^{iS_{MSR}} \underset{\text{reverse}}{=} \int_{\substack{x(-t)=z \\ x(0)=y}} Dx\, D\pi\, e^{iS_{MSR} + \underbrace{\int_{-t}^{0} dt \frac{d\Phi}{dt}}_{\text{offset}}}$$

$$= \langle y| e^{-\hat{W}t} |z\rangle\, e^{\Phi(y)}\, e^{-\Phi(z)}$$

Detailed balance: $\langle z| e^{-\hat{W}t} |y\rangle e^{-\Phi(y)} = \langle y| e^{-\hat{W}t} |z\rangle e^{-\Phi(z)}$

- conserved quantities (strong symmetry):

$\frac{dF}{dt} = 0$ before noise average. $\Rightarrow H_{MSR}(\pi_i + \partial_i F) = H_{MSR}(\pi_i)$

Example 1: Johnson noise. Consider capacitor holding charge q.

thermal equilibrium: $\Phi = \beta \cdot \frac{q^2}{2C}$

Time-reversal: $q(t) \longmapsto q(-t)$, $\quad \pi \to -\pi(-t) + i\mu$

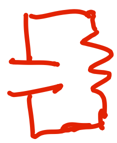
$$L = \pi \dot{q} + i\pi \cdot (\pi - i\mu) \cdot \Gamma$$

swap under T

← phenomenological constant.

What are the noise-free equations?

$$\left.\frac{\delta S_{MSR}}{\delta \pi}\right|_{\pi=0} = 0 = \dot q + \Gamma \mu = \dot q + \Gamma \frac{\beta q}{C}$$

Interpret:  RC circuit $\quad R = \frac{1}{\beta \Gamma}$

usually: $\Gamma \sim$ current noise $\sim \frac{1}{T \cdot R}$ (Johnson noise)

Example 2: Spin dynamics. $\quad S_i(t) = (S_x, S_y, S_z)$.

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k .$$

Look for thermal $\Phi = \beta H$ where $H = -h_i S_i = -\vec h \cdot \vec S$ $\hookrightarrow \mu_i = -\beta h_i$

Dissipationless dynamics: $L_{MSR} = \pi_i \dot S_i - H_{MSR}(\pi, S) \to = \pi_i C_i(S)$

Under T: $(S \to S)$:

$$\pi_i C_i \to (-\pi_i + i\mu_i) C_i = \pi_i C_i^{rev} \quad \text{(no } \pi\text{-ind.)}$$

Choose: $C_i = -\varepsilon_{ijk} h_j S_k \to$ contract k!

orthogonal to h.

Equation: $\left.\frac{\delta S}{\delta \pi}\right|_{\pi=0} = \dot S_i - C_i = 0$ or $\dot S_i = -\varepsilon_{ijk} h_j S_k$

$$\dot{\vec S} = -\vec h \times \vec S$$

$$\dot S_i = \{S_i, S_j\} \frac{\partial H}{\partial S_j} = \{S_i, H\}$$

(dissipationless)

Therefore $L_{MSR} = \pi_i \dot S_i - \frac{1}{\beta} \pi_i \overbrace{\{S^i, S^j\}}^{\text{antisymmetric } Q} \mu_j$, $\quad \mu_j \to \beta \frac{\partial H}{\partial S_j}$

general result: (Hamiltonian) $\pi_i \{q_i, q_j\} \mu_j / \beta \ldots$

This problem has "strong" symmetry or constraint: $\frac{d}{dt}\vec{S}\cdot\vec{S}=0$.

$$\mathcal{H}(\pi_i)=\mathcal{H}(\pi_i+2S_i)$$

$$L_{MSR} = \pi_i \dot{S}_i - (\pi_i + \cancelto{0}{2S_i})\varepsilon_{ijk} h_j S_k$$

Dissipative dynamics obeying constraint?

Invariant building block under T & constraint?

Under T:

$h \to +h$

$S \to +S$

$\pi \to -\pi + i\mu$

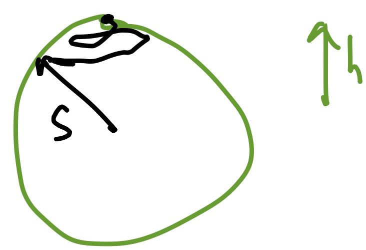
Constraint: $\pi_i \varepsilon_{ijk} S_j [\cdots]_k$

(temp)

$$T\gamma \cdot (\pi_i \varepsilon_{ijk} S_j) \cdot i\left((\pi_m - i\mu_m)\varepsilon_{mnk} S_n\right).$$

Result: $\left.\frac{\delta S}{\delta \pi}\right|_{\pi=0} = 0 \rightsquigarrow \dot{\vec{S}} = \vec{h}\times\vec{S} - \underbrace{\gamma\,\vec{S}\times(\vec{h}\times\vec{S})}_{} + \text{noise}$

Landau-Lifshitz Gilbert damping



Now to turn to field theory:

DOF: $\underbrace{\phi(x}_{q_i \text{ or } x_i};t)$

$L_{MSR} = \int dx\, \mathcal{L}$

$$\mathcal{L}_{MSR} = \pi\partial_t\phi - \mathcal{H}$$

$\mathcal{H}(\pi=0)=0$, $\mathrm{Im}(\mathcal{H}) \leq 0 \ldots$

Fokker-Planck:

$$\frac{\partial P}{\partial t} = \text{"}\frac{\delta}{\delta\phi}\left[Q[\phi,\phi']\frac{\delta P}{\delta \phi'}\right]\text{"}$$

!!

Example 3: dissipative Klein-Gordon theory

Start with: $\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}c^2(\partial_x\phi)^2 - \frac{1}{2}m^2\phi^2$

Dissipative $\mathcal{L}_{MSR}$ route through Hamiltonian field theory:

Define conjugate momentum: $\psi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi$

$\mathcal{H} = \psi\partial_t\phi - \mathcal{L} = \frac{\psi^2}{2} + c^2(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2$. $\{\phi(x_1),\psi(x_2)\} = \delta(x_1-x_2)$

Consider thermal state $\Phi = \beta H = \beta\int dx\,\mathcal{H}$

$\{\phi,\phi\} = \{\psi,\psi\} = 0$

Dissipationless MSR:

$$L_{MSR} = \int dx\, \mathcal{L}_{MSR} = \int dx\left[\overset{\text{noise}}{\pi}\partial_t\phi + \sigma\partial_t\psi\right]$$

$$-\int dx_1 dx_2\left[\pi(x_1)\{\phi(x_1),\psi(x_2)\}\frac{\delta H}{\delta\psi(x_2)} + \sigma(x_1)\{\psi(x_1),\phi(x_2)\}\frac{\delta H}{\delta\phi(x_2)}\right]$$

$$\mathcal{L}_{MSR} = \pi\partial_t\phi + \sigma\partial_t\psi - \pi\psi + \sigma\left[m^2\phi - c^2\partial_x^2\phi\right]$$

Dissipationless EOM: $\partial_t\phi = \psi$

$\partial_t\psi = -m^2\phi + c^2\partial_x^2\phi = \partial_t^2\phi$

Now add dissipation!

$(\gamma > 0)$ $(\eta > 0)$

$$\mathcal{L}_{MSR} \to \mathcal{L}_{MSR} + i\gamma\pi(\pi - i\mu_\phi) + i\eta\sigma(\sigma - i\mu_\psi)$$

$$+ i\gamma\pi(\pi - i\beta m^2\phi + i\beta c^2\partial_x^2\phi) + i\eta\sigma(\sigma - i\beta\psi)$$

For simplicity, take $\gamma = 0$.

Equations: $\partial_t \phi = \psi$

$$\partial_t \psi = -m^2 \phi + c^2 \partial_x^2 \phi - \eta \beta \psi$$

$$\downarrow$$

$$\partial_t^2 \phi = -m^2 \phi + c^2 \partial_x^2 \phi - \eta \beta \partial_t \phi$$

Look for quasi-normal modes: $\phi = e^{ikx - i\omega t}$; $\omega(k)$?

$$0 = +\omega^2 - m^2 - c^2 k^2 + i\omega \eta \beta$$

$$\omega = \pm\sqrt{m^2 + (\eta\beta)^2 + c^2 k^2} - \frac{i\eta\beta}{2}$$

$\eta > 0$

$\beta > 0$: expanding

= quasi!

$\mathrm{Im}(\omega) \leq 0 \rightsquigarrow$ systems relax towards equilibrium ($\phi = 0$)

$$\phi \sim e^{-\eta\beta t/2} \quad \text{NOT} \quad e^{+\beta\eta t/2}.$$