PHYS 7810
Hydrodynamics
Spring 2024

Lecture 6
Diffusion

February 1

Hydrodynamics = effective field theory of relaxation of conserved quantities to (thermal) equilibrium

MSR field theory

strong symmetry

Simplest model of hydro = diffusion of one conserved quantity

$$Q = \int dx \, \rho(x)$$

Write down MSR Lagrangian

$$\mathcal{L} = \pi \partial_t \rho - \mathcal{H}_{MSR}(\pi, \rho) \rightarrow \pi \# + \pi^2 \# + \cdots$$

Conservation of Q: $\mathcal{H}_{MSR}$ must be invariant under

$$\pi \rightarrow \pi + \varepsilon \frac{\delta Q}{\delta \rho(x)} = \pi + 1 \qquad \left[\frac{\delta Q}{\delta \rho(x)} = \int dy \, \underbrace{\frac{\delta \rho(y)}{\delta \rho(x)}}_{\delta(x-y)} = 1 \right]$$

Invariant building blocks: $\partial_x \pi$ $\quad [\pi(x+\delta x) - \pi(x)]$

$\hookrightarrow$ so $\mathcal{H}_{MSR} = (\partial_x \pi)\cdot J$

Assume microscopic time-reversal symmetry:

$$\mathcal{H}_{MSR}(\pi, \rho) = \mathcal{H}_{MSR}(-\pi + i\mu, \rho)$$

[assume $\rho \to \rho$ under T]

What was μ? $\quad \mu = \dfrac{\delta \Phi}{\delta \rho(x)} \quad [P_{ss} = e^{-\Phi}]$

For this problem:

$$\Phi = \beta \int dx \, \frac{\rho(x)^2}{2\chi} \quad \text{where} \quad \mu = \frac{\beta}{\chi}\rho(x) = \beta \underbrace{\mu_{th}}_{\text{chemical potential}}$$

where $\mu_{th}\chi = \rho$

charge susceptibility $\rightarrow \chi = \dfrac{\partial \rho}{\partial \mu}$

$$\mathcal{L}_{MSR} = \pi \partial_t \rho + \underbrace{\partial_x \pi \, \partial_x (\pi - i\mu)}_{T\text{-even}} \cdot iT\underbrace{\sigma}_{>0}$$

(charge) conductivity: σ

to ensure $-\mathrm{Im}(\mathcal{L}_{MSR}) = \mathrm{Im}(\mathcal{H}) \leq 0$.

Noise-free equations of motion:

$$\left.\frac{\delta S}{\delta \pi}\right|_{\pi=0} = \partial_t \rho - \partial_x\left[iT\sigma \partial_x(\cancel{\pi} - i\mu)\right] = 0$$

$$\partial_t \rho = \partial_x(\cancel{T}\sigma \cancel{\beta}\, \partial_x \mu_{th}) = D\partial_x^2 \rho$$

$\nwarrow$ if σ const.

D = diffusion const.

$= \dfrac{\sigma}{\chi}$

$\uparrow$

Einstein relation

Add noise back: (stochastic) fluctuating hydrodynamics

$$\partial_t \rho = D\partial_x^2 \rho + \xi(x,t) \quad \text{where} \quad \langle \xi \rangle = 0$$

$$\langle \xi(x,t)\xi(x',t')\rangle = 2T\sigma \cdot -\partial_x^2 \delta(x-x')$$

Re-write equation as conservation law:

$$\partial_t \rho + \partial_x J = 0$$

current (spatial part)

$$\frac{dQ}{dt} = \int dx \frac{\partial \rho}{\partial t} = \int dx \left(-\frac{\partial J}{\partial x}\right) \to 0$$

Write a constitutive relation:

$$J = -D\,\partial_x \rho + \xi_J(x,t) \quad \text{where} \quad \langle \xi_J \rangle = 0, \quad \langle \xi_J(x,t)\,\xi_J(x',t') \rangle = \underbrace{2T\sigma}\cdot\delta(x-x')$$

conductivity sets noise str:
fluctuation-dissipation thm.

$$[J] = \frac{\cancel{[\rho]}[L]}{[T]} \qquad [\xi_J^2] = [en]\cdot[\sigma]\,[L]^{-1}$$

$$\frac{[en]}{[L]} = [E] = [J]/[\sigma]$$

$\rho \rightsquigarrow$ electric charge
$\mu_{th} \rightsquigarrow$ (electro) chemical potential
$\partial_x \mu_{th} \to -E$ (electric field)

$$\langle J \rangle = -D\,\partial_x(\chi \mu_{th}) = \sigma E$$

Ohm's Law!

$$\partial_t \rho = D\,\partial_x^2 \rho$$ solution was in lec 2

$$\rho(x,t) = \int dy\, \rho(y,0)\, \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi D t}}$$

FPE of one particle
↓
linear

lumps of charge spread on $\Delta x \sim \sqrt{Dt}$

Physical interpretation here different:

Q = macroscopic total charge

$\rho(x,t) \sim$ many particles (many-body)

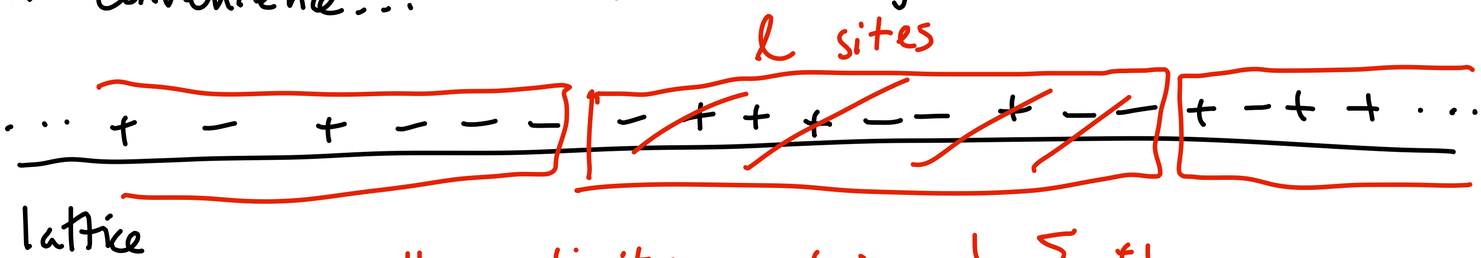
↓

$$\partial_t \rho = \partial_x\left(D(\rho)\,\partial_x \rho\right)$$

Why $\Phi = \int dx \, \beta \cdot \frac{\boxed{\rho^2}}{2\chi}$ ⇝ Φ is Gaussian?

Postulate: in thermodynamic state, "ergodic hypothesis" says that all states w/ same Q equally likely...

For convenience... $Q = 0$ [pos/neg charge cancel overall]



$\xrightarrow[\text{hydro}]{}$ continuum limit: $\rho(x) \sim \frac{1}{\ell} \sum_{\text{sites}} \pm 1$

In any fixed box: (lec 1)

$$\mathbb{P}[m \text{ "+"}, \ell - m \text{ "−"}] = \binom{\ell}{m} \frac{1}{2^\ell} \sim \underbrace{\exp\left[- \frac{(\ell/2 - m)^2}{2 \, \ell/4} \right]}_{\text{Gaussian (central limit thm)}}$$

Coarse grain: $\rho(x)$ typically of order $\frac{1}{\sqrt{\ell}}$, $\rho(x_1)$ & $\rho(x_2)$ $\approx$ uncorrelated in thermo limit

$$\Phi = -\log P_{ss} = \sum_{\text{box}} \Phi_{\text{box}} \sim \sum \frac{(\ell/2 - m_{\text{box}})^2}{\ell/2} \to \frac{\beta}{\chi} \int dx \, \frac{\rho^2}{2}$$

If only positive:

$$\beta \int dx \, \frac{1}{2} \frac{(\rho - \overset{\text{avg density}}{\bar{\rho}})^2}{\chi} = \beta \int dx \left[\frac{\rho^2}{2\chi} - \overbrace{\frac{\bar{\rho}}{\chi} \cdot \rho}^{= -\beta \frac{\bar{\rho}}{\chi} Q} + \overbrace{\cancel{\frac{\bar{\rho}^2}{2\chi}}}^{\text{const}} \right]$$

Another lesson (lec 7): hydrodynamics = EFT on long length scales ($\ell \to \infty$) compared to microscopic mean free path

EFT for diffusion without time-reversal symmetry.

- Our thermodynamics (Φ) still reasonable...

Per Lec 3 $\mapsto$ exists a "reversed" stochastic process

↳ $\mathcal{L}^{rev}_{MSR}$ must be another valid MSR theory

$$= \pi \partial_t \rho - \mathcal{H}(-\pi + i\mu, \rho) = \pi \partial_t \rho - \mathcal{H}^{rev}(\pi, \rho)$$

↑ no π^0 terms!

What can we add?

$$\mathcal{L}_{MSR} = \pi \partial_t \rho + \underbrace{i T\sigma \partial_x \pi \, \partial_x(\pi - i\mu)}_{\text{T-even}} + \underbrace{\alpha \partial_x \pi \cdot f(\mu)}_{\text{T-odd}} + \cdots$$

T-odd: $\alpha \partial_x(-\pi + i\mu) \cdot f(\mu) = \underset{\text{T-odd...}}{-\alpha \partial_x \pi f(\mu)} + \cancel{i\alpha \underbrace{\partial_x \mu f(\mu)}}$

$$= \frac{d}{dx} F(\mu)$$

i.e., offset is total derivative

$$\frac{dF}{d\mu} = f$$

What are consequences? noise-free EOMs

$$\left.\frac{\delta S}{\delta \pi}\right|_{\pi=0} = \partial_t \rho - \underbrace{\partial_x(T\sigma \partial_x \mu) - \alpha \partial_x f(\mu)} = 0$$

constitutive: $J = \underbrace{-T\sigma \partial_x \mu}_{\text{Fick's Law}} - \alpha f(\mu)$

Taylor expand: $f(\mu) = \cancel{f_0} - \frac{v}{\alpha}\rho + \cdots$

↑ const.

Resulting EoM: $\partial_t \rho = -v\partial_x \rho + D\partial_x^2 \rho$ (biased diffusion)

<u>Quasinormal modes</u>: $\rho \sim e^{ikx - i\omega t}$

$$0 = \underbrace{\left[-i\omega + ikv + Dk^2\right]}_{\text{dispersion relation}} e^{ikx - i\omega t}$$

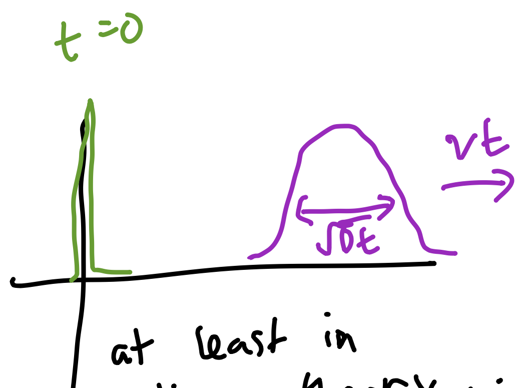
$$\omega(k) = \underbrace{v\cdot k}_{\mathrm{Re}(\omega)\neq 0} \underbrace{-iDk^2}_{\mathrm{Im}(\omega)\leq 0}$$

$\mathrm{Re}(\omega) \neq 0$: "ballistic"

$\mathrm{Im}(\omega) \leq 0$: <u>dissipative</u>: $\rho \sim e^{-Dk^2 t}$

Re-visit exact solution:

$$\rho(x,t) = \int dy\, \rho(y,0)\, \frac{e^{-(x-vt-y)^2/4Dt}}{\sqrt{4\pi D t}}$$



at least in linear theory...