PHYS 7810
Hydrodynamics
Spring 2024

Lecture 7
Hydrodynamic correlation functions

February 6

Hydro EFT of diffusion: n conserved charges, $(a=1,\dots,n)$
d spatial dimensions. $(i,j=1,\dots d)$

$$\mathcal{L}_{MSR} = \pi^a \partial_t \rho^a - \mathcal{H}(\pi^a, \rho^a)$$

↳ invariant under $\pi^a \to \pi^a + c^a$ (conservation laws)

const. vector

Postulate Gaussian thermodynamic fluctuations:

$$\Phi = \int dx \ \frac{\beta}{2} \rho^a (\chi^{-1})^{ab} \rho^b \qquad \mu_b = \frac{\delta \Phi}{\delta \rho^b}$$

Impose time-reversal symmetry:

$$\mathcal{L}_{MSR} = \pi^a \partial_t \rho^a + i T \sigma^{ab} \partial_i \pi^a \partial_i (\pi^b - i\mu^b)$$

↓ equations of motion:

D^{ab} = diffusion const. matrix.

$$\frac{\delta S}{\delta \pi^a} = 0 = \partial_t \rho^a - \partial_i \Big(\overbrace{\sigma^{ac} (\chi^{-1})^{cb}}^{D^{ab}} \partial_i \rho^b \Big)$$

How do we detect diffusion?

Correlation functions! Take local observable $A(x)$:

Classical: $\langle A(x,t)\, A(y,0)\rangle \rightsquigarrow$ hydro poles in Fourier trans.

quantum: $\langle \{A(x,t), A(y,0)\}\rangle = G^S_{AA}(x-y,t)$

$\langle [A(x,t), A(y,0)]\rangle \cdot i\Theta(t) = G^R_{AA}(x-y,t)$

Use MSR path integral to calculate G^S (and G^R):

$$G^S_{\rho\rho}(k,\omega) = \int D\pi D\rho\, \underbrace{e^{iS_{MSR}}}\, \rho(k,\omega)\, \rho(-k,-\omega)$$

$$= \exp\left[-\int dk\, d\omega\, (\rho \;\; \pi)\, \frac{M}{2} \begin{pmatrix} \rho \\ \pi \end{pmatrix}\right]$$

where $M = \begin{pmatrix} & -\omega - iDk^2 \\ -\omega + iDk^2 & 2T\sigma k^2 \end{pmatrix}$ where $D = \frac{\sigma}{\chi}$

Doing integral: $G^S_{\rho\rho} = (M^{-1})_{\rho\rho} = \dfrac{2T\sigma k^2}{\omega^2 + (Dk^2)^2}$

highly singular as $k, \omega \to 0$.

$\lim_{\omega\to 0}\lim_{k\to 0} \neq \lim_{k\to 0}\lim_{\omega\to 0}$.

More physically transparent: retarded Green's function $G^R_{\rho\rho}$:

$$G^R(k,\omega) \sim C \cdot (M^{-1})_{\rho\pi} \sim \langle \rho\pi\rangle \sim C \cdot \frac{-1}{\omega + iDk^2}$$

pole at $\underline{\omega = -iDk^2}$.

Interpretation: (heuristic)

take $\Phi \to \Phi - \beta \cdot \varepsilon\, \rho(k,\omega)\, e^{ikx - i\omega t}$, thus

$\mu \to \mu - \varepsilon e^{ikx - i\omega t}$ β

Then $\mathcal{L}_{MSR} \to \mathcal{L}_{MSR} + \underbrace{\sigma \partial_x \pi\, \partial_x(-\varepsilon e^{ikx - i\omega t})} \to \sigma k^2 \pi e^{ikx - i\omega t} \varepsilon$

Now evaluate:

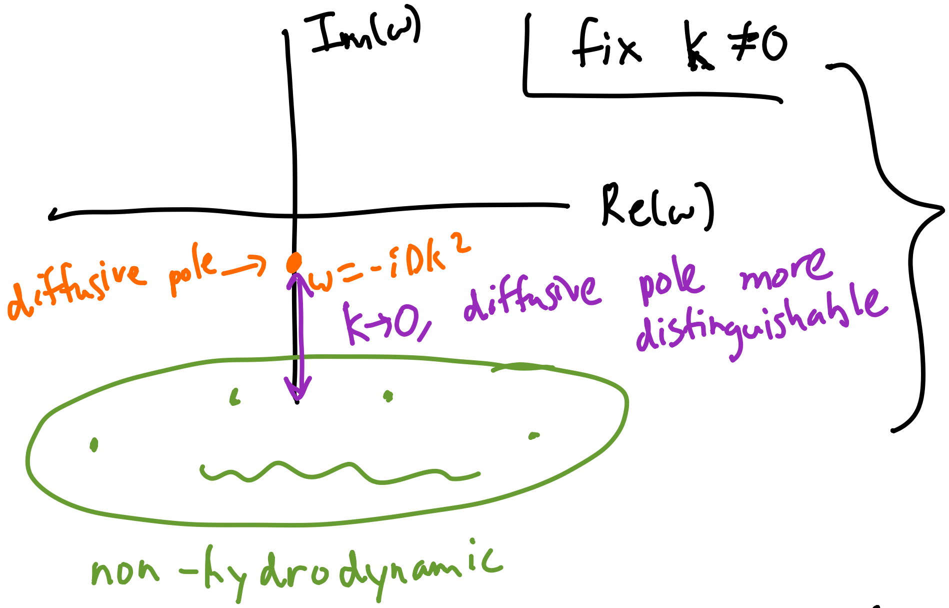
$$\langle \rho \rangle = \int D\pi D\rho \, e^{iS_{MSR} + i\varepsilon\sigma k^2 \pi} \rho \quad \text{to first order in } \varepsilon$$

$$\sim \int D\pi D\rho \, e^{iS_{MSR}} \, \pi\rho \cdot \varepsilon \boxed{i\sigma k^2} \to C$$

Thus: $G^R_{\rho\rho} = \chi \dfrac{Dk^2}{Dk^2 - i\omega}$

diffusive quasinormal mode.

Pole of G^R [denom = 0] is $\boxed{\omega = -iDk^2}$



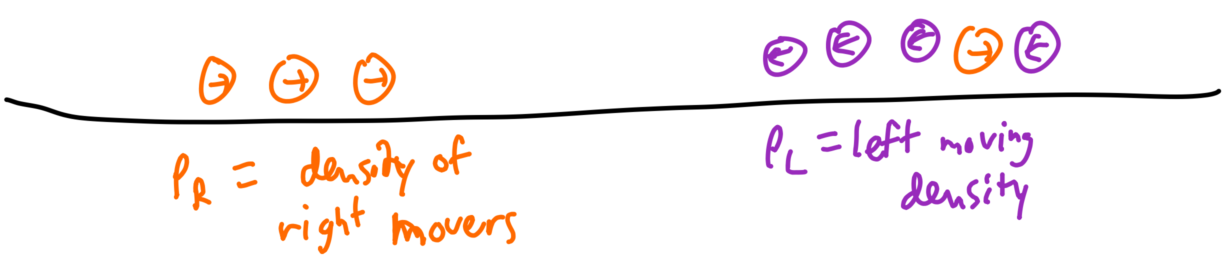
structure of $G^R(\omega)$ in complex-ω plane.

"general" operators A!
G^R_{AA} have diffusive pole...

Why useful? Fourier transform $G^R(\omega) \to G^R(t)$

$$G^R(k,t) \sim e^{-Dk^2 t} + e^{-t/t_{\text{not hydro}}} + \ldots$$

Model of hydro & non-hydro modes:



Equations of motion:

$$\left.\begin{aligned} \partial_t \rho_L - v\partial_x \rho_L &= -\gamma(\rho_L - \rho_R) \\ \partial_t \rho_R + v\partial_x \rho_R &= -\gamma(\rho_R - \rho_L) \end{aligned}\right\}$$

"collision" try to equalize $\rho_L = \rho_R \ldots$

$\partial_t(\rho_L + \rho_R) + \partial_x J = 0.$

$J = v(\rho_R - \rho_L)$

Solve EOMs! (in Fourier space): $e^{ikx-i\omega t}\begin{pmatrix} a_L \\ a_R \end{pmatrix} = \begin{pmatrix} \rho_L \\ \rho_R \end{pmatrix}$:

$$-i\omega\begin{pmatrix} a_R \\ a_L \end{pmatrix} + \begin{pmatrix} \gamma + ikv & -\gamma \\ -\gamma & \gamma - ikv \end{pmatrix}\begin{pmatrix} a_R \\ a_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

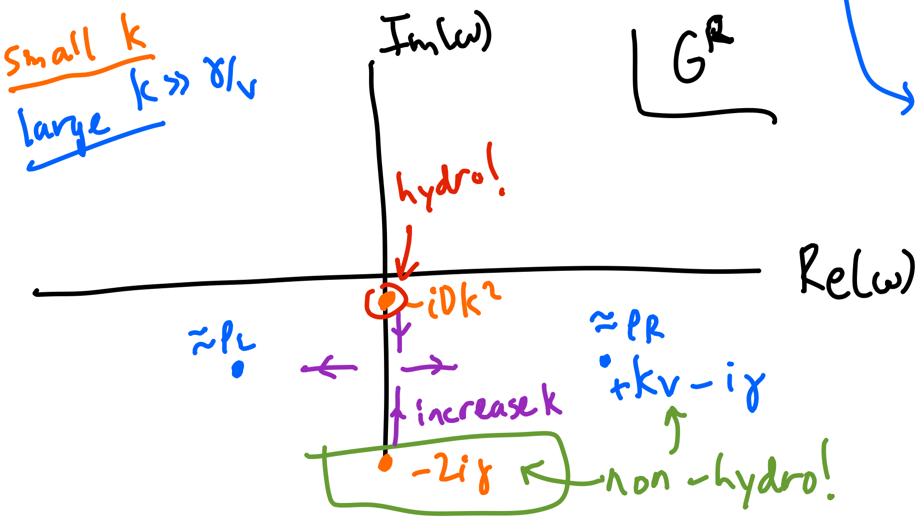
QNM ω are eigenvalues of 2×2 matrix!

$$\omega = -i\left[\gamma \pm \sqrt{\gamma^2 - (vk)^2}\right]$$

$\omega \approx -2i\gamma$ or $-i\frac{v^2}{2\gamma}k^2 = -iDk^2$

G^R

$\omega \approx -i\gamma \pm kv$



lessons: - microscopic model contains non-hydro modes, visible at $|\omega| \sim \gamma$, or $\Delta t \sim \frac{1}{\gamma}$.

"mean free time" between collisions.

- for $\Delta t \gg 1/\gamma \to$ hydro!
- corrections to diffusion!

$$\omega = -i\left[\gamma - \sqrt{\gamma^2 - (kv)^2}\right] = -i\gamma\left[\frac{1}{2}\left(\frac{vk}{\gamma}\right)^2 + \frac{1}{8}\left(\frac{vk}{\gamma}\right)^4 + \cdots\right]$$

$$= -iDk^2\left[1 + (\ell k)^2 + \cdots\right]$$

where $\ell = \frac{v}{2\gamma} =$ "mean free path": length scale below which hydro doesn't exist.

Hydro as <u>effective field theory</u>: makes sense for lengths $\gg \ell$, time $\gg 1/\gamma$

Justify hydro as EFT considering "scaling theory":

$$S_{MSR} = \int dt\, d^dx \left[\underbrace{\pi \partial_t \rho} + \underbrace{i T \sigma \partial_i \pi \partial_i (\pi - i\mu)} + \cdots \right]$$

ignored until now — why?

Idea: $\vec{x} \to \dfrac{\vec{x}}{\lambda} = \lambda^{-1}\vec{x}$ $\quad [\vec{x}] = -1$ "scaling dimension".

$t \to \lambda^{[t]} t$, $\quad \pi \to \lambda^{[\pi]}\pi \quad \rho \to \lambda^{[\rho]}\rho \ldots$

$$\left([\pi] - \cancel{[t]} + [\rho]\right) + \left(\cancel{[t]} + d[x]\right) = 0$$

respect T: $[\pi] = [\mu] = [\rho]$

$\Big\}\ [\pi] = [\rho] = d/2.$

$$\left(2[\pi] - 2[x]\right) + \left([t] + d[x]\right) = 0$$

$\Big\}\ [t] = -2.$

$\downarrow$

$t \to t/\lambda^2$ or

x^2/t invariant...

Go to long length scales $(\lambda \to 0)$ to see diffusion.

Other terms in MSR action?

$$S_{MSR} = \cdots + \underbrace{\int dt\, d^dx \;\; i\eta\, \partial_x^m \pi\, \partial_x^m (\pi - i\mu) \cdot \rho^n}_{A}$$

A: $[A] = 2(m-1) + \frac{d}{2} n \geq 0.$

Need $m \geq 1$ (charge cons. $\pi \to \pi + 1$ symmetry) & $n \geq 0$.

Under rescaling $\quad A \to \lambda^{[A]} A \to 0 \quad$ if $\lambda \to 0$.

Ignore A on long length scales.

$A \sim$ "irrelevant".

Only irrelevant corrections to diffusive MSR $\to$ diffusive hydro is <u>stable</u>.

What if we break time-reversal?

$$S_{MSR} = \cdots + \int dt\, d^d x\ \partial_x \pi \left(\alpha_1 \rho + \alpha_2 \rho^2 + \cdots \right)$$

dimension: -1 "relevant"

(but undo α_1: $x \to x - \alpha_1 t$)

dimension: $\frac{d-2}{2}$

↑ relevant if $d=1$!

At long length scales, α_2 term becomes MORE important...

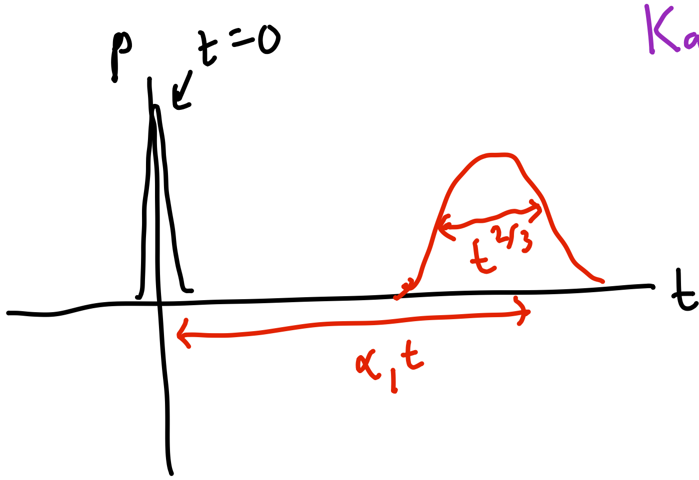
Try to find "new scaling theory":

$$[\pi \partial_t \rho] = [\partial_x \pi \cdot \rho^2] \quad \text{and} \quad [\pi] = [\rho]$$
$$= +1 - [t]$$

So $[\pi] + [\rho] = 1$ and $2[\pi] - [t] = 3[\pi] - 1$

find $[\pi] = 1/2$ and $\boxed{[t] = -3/2.}$

Kardar-Parisi-Zhang



Scaling observable in numerics!