PHYS 7810
Hydrodynamics
Spring 2024

Lecture 8
Navier-Stokes equations

February 8

$\underbrace{\text{Hydrodynamics}}$ of liquids & gases

dissipative EFT; degrees of freedom = $\underbrace{\text{conserved quant.}}$

(M) Particles / Mass / "charge" (ρ)

(P_i) Momentum (g_i)

(E) Energy (ε)

(L_{ij}) Angular Momentum (return in lec 9)

Statistical stationary state?

$$P_{ss} = e^{-\Phi} = e^{-\beta E} \rightsquigarrow \underbrace{e^{S}}_{\text{count "field configs" available...}} \quad \text{where} \quad \underset{\text{total entropy}}{S} = \int \underbrace{d^d x}_{d \text{ spatial dims.}}\, s(\varepsilon, \rho, g_i)$$

Thermodynamic review!

First law of thermo: $(k_B = 1)$

$$T\,ds = d\varepsilon - \mu_{th}\,d\rho - v_i\,dg_i \rightarrow \left.\frac{\partial s}{\partial \varepsilon}\right|_{\rho, g} = \frac{1}{T}$$

$$ds = \frac{1}{T}d\varepsilon - \frac{\mu_{th}}{T}d\rho - \underbrace{\frac{v_i}{T}}dg_i$$

thermo conjugate: $-v_i/T$:

$$\mu_\varepsilon = -\frac{1}{T} \qquad \mu_\rho = \frac{\mu_{th}}{T} \qquad \mu_{g_i} = \frac{\delta\Phi}{\delta g_i} = -\frac{\delta S}{\delta g_i} = \frac{v_i}{T}$$

Gibbs-Duhem relation:

$$dE = TdS + \mu_{th}dM + v_i dP_i - \underbrace{PdV}_{\text{work done by system}\to\text{environ.}}$$

$$\frac{dE}{dV} = \boxed{\varepsilon = Ts + \mu_{th}\rho + v_i g_i - P}$$

General MSR Lagrangian $\mathcal{L}_{MSR}(\varepsilon, \rho, g_i, \pi_\varepsilon, \pi_\rho, \pi_{g_i} = \pi_i)$?

Time-reversal symmetry:

$$\rho \to \rho \qquad \varepsilon \to \varepsilon \qquad g_i \to -g_i$$

$$\pi_\rho \to -\pi_\rho + i\mu_\rho \qquad \pi_\varepsilon \to -\pi_\varepsilon + i\mu_\varepsilon \qquad \pi_i \to \pi_i - i\mu_i$$

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_\varepsilon \partial_t \varepsilon + \pi_i \partial_t g_i - \mathcal{H}_{MSR}(\cdots)$$

$$\mathcal{H}_{MSR} = (\partial_i \pi_\rho) J^i + (\partial_i \pi_\varepsilon)\mathcal{E}^i + (\partial_i \pi_j)\tau_{ij}$$

due to conservation laws of M, E, P_j.

"mass/charge" current

stress tensor

What are constitutive relations, i.e.

$$J_i = -D\partial_i \rho + \cdots \quad ?$$

Note: $\left.\frac{\delta S_{MSR}}{\delta \pi_\rho}\right|_{\pi_\rho=0} = \partial_t \rho - \partial_i(-J^i) = \overset{-\,+}{\partial_t \rho} + \overset{+\,-}{\partial_i J_i} = 0$

local conservation law

First look for ideal hydro (dissipationless & $J_i(\rho, \varepsilon, g)$ ind. of derivatives):

Time-reversal symmetry:

$J_i \to -J_i \qquad \mathcal{E}_i \to -\mathcal{E}_i \qquad \tau_{ij} \to \tau_{ij}$

Assume rotation invariance...:

$\mathcal{E}_i = A(\varepsilon, \rho, g_i g_i)\, v_i \,; \qquad J_i = B \cdot v_i \,; \qquad \tau_{ij} = C\delta_{ij} + D v_i v_j$

But impose T-even:

$$\mathcal{L}_{MSR} \to \mathcal{L}_{MSR} + i\Big[\underbrace{\mu_\varepsilon \partial_t \varepsilon + \mu_\rho \partial_t \rho + \mu_i \partial_t g_i}_{\text{detailed balance, or thermo: } -\partial_t s} - \underbrace{\mathcal{E}_i \partial_i \mu_\varepsilon - J_i \partial_i \mu_\rho - \tau_{ij}\partial_i \mu_j}_{\text{total divergence? } -\partial_i s^i}\Big]$$

$$= A v_i \partial_i \frac{1}{T} - B v_i \partial_i\left(\frac{\mu_{th}}{T}\right) - \underbrace{C\partial_i\left(\frac{v_i}{T}\right)}_{= \underbrace{\partial_i\left(\frac{C}{T} v_i\right)}_{\text{total div}} - \frac{v_i}{T}\partial_i C} - D v_i v_j \partial_i\left(\frac{v_j}{T}\right) = \underbrace{\partial_i S_i}_{\text{total div}} ?$$

$C(\varepsilon, \rho, g)$

C is $C\left(\frac{1}{T}, \frac{\mu_{th}}{T}, \frac{v_j v_j}{2T}\right)$:

$$\partial_i C = \left.\frac{\partial C}{\partial(1/T)}\right|_{\frac{\mu_{th}}{T}, \frac{v^2}{T}} \partial_i\left(\frac{1}{T}\right) + \left.\frac{\partial C}{\partial(\mu_{th}/T)}\right|_{1/T, v^2/T} \partial_i \frac{\mu_{th}}{T} + \left.\frac{\partial C}{\partial\left(\frac{v^2}{2T}\right)}\right|_{\frac{1}{T}, \mu} \underbrace{\partial_i \frac{v^2}{2T}}_{= \frac{v^2}{2}\partial_i \frac{1}{T} + \frac{v_j}{T}\partial_i v_j}$$

Choose A, B, D to cancel $\partial_i C$:

$$0 = A v_i + \left.\frac{\partial C}{\partial(1/T)}\right|_{\mu/T,\, v^2/T} \cdot \frac{v_i}{T} - D v_i \frac{v_j v_j}{T} + \left.\frac{\partial C}{\partial\left(\frac{v^2}{2T}\right)}\right|_{\frac{1}{T},\frac{\mu}{T}} \frac{v_i v_j v_j}{2T}$$

$$0 = -B v_i + \frac{v_i}{T} \left.\frac{\partial C}{\partial(\mu/T)}\right|_{1/T,\, v^2}$$

$$0 = \frac{v_i v_j}{T} \left.\frac{\partial C}{\partial(v^2/2T)}\right|_{T,\mu} - D\frac{v_i v_j}{T}.$$

remind of thermo!

A, B, D fixed by one function $C(\varepsilon, \rho, g_i)$.

define to be pressure P.

$$D = \frac{1}{T} \left.\frac{\partial P}{\partial\left(\frac{v^2}{2T}\right)}\right|_{\frac{1}{T}, \frac{\mu_{th}}{T}} = \left.\frac{\partial P}{\partial(v^2/2)}\right|_{T,\mu}$$

Use Gibbs-Duhem & 1st law:

$$d(\varepsilon + P) = d\varepsilon + dP = d(\mu_{th}\rho + v_i g_i + Ts)$$

$$dP = s\,dT + \rho\, d\mu_{th} + g_i dv_i$$

$$\frac{\partial P}{\partial v_i} = g_i, \quad \text{then} \quad \frac{\partial P}{\partial v_i} = v_i \frac{\partial P}{\partial(v^2/2)} \cdots$$

$\underbrace{g_i = D v_i}$, "mass density" for Gal. fluid (lec 9)

Next: $$B = \frac{1}{T} \left.\frac{\partial P}{\partial\left(\frac{\mu_{th}}{T}\right)}\right|_{\frac{1}{T}, v^2/T} = \left.\frac{\partial P}{\partial \mu_{th}}\right|_{T,v^2} = \rho$$ $B = \rho$

Similar calculation: $A = \varepsilon + P$

Combine, constitutive relations for ideal fluid:

$$\mathcal{E}_i = (\varepsilon + P) v_i \qquad J_i = \rho v_i \qquad \tau_{ij} = P\delta_{ij} + D v_i v_j$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\partial_t \varepsilon + \partial_i((\varepsilon + P)v_i) = 0 \qquad \partial_t \rho + \partial_i(\rho v_i) = 0 \qquad \partial_t g_i + \partial_i P + \partial_j(D v_i v_j) = 0.$$

Euler eq.

What are noise & dissipation addable to EFT?

- respect T
- add as few ∂_i as possible. (lec 7: hydro on long wavelengths)

For momentum cons:

$$\mathcal{L}_{MSR} = \cdots + i\partial_i \pi_j T\eta_{ijkl}\,\partial_k(\pi_l - i\mu_l)$$

viscosity tensor: rotation invariance:

$$\eta_{ijkl} = \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\right) + \zeta\,\delta_{ij}\delta_{kl}$$

(shear) (bulk)

For energy & mass/charge:

$$\mathcal{L}_{MSR} = \cdots + iT\partial_i\begin{pmatrix}\pi_\varepsilon & \pi_\rho\end{pmatrix}\underbrace{\begin{pmatrix}A^{\varepsilon\varepsilon} & A^{\varepsilon\rho} \\ A^{\varepsilon\rho} & A^{\rho\rho}\end{pmatrix}}_{\text{incoherent conductivity}}\partial_i\begin{pmatrix}\pi_\varepsilon - i\mu_\varepsilon \\ \pi_\rho - i\mu_\rho\end{pmatrix}$$

constitutive relations:

$$J_i = \rho v_i - TA^{\rho\rho}\partial_i\left(\frac{\mu_{th}}{T}\right) - TA^{\varepsilon\rho}\partial_i\left(\frac{1}{T}\right)$$

$$= \rho v_i - \sigma\partial_i\mu_{th} - \underline{\alpha}\,\partial_i T$$

Onsager reciprocity

$$\mathcal{E}_i = (\varepsilon + P)v_i - \underline{T\alpha}\,\partial_i\mu_{th} - \bar{\kappa}\,\partial_i T$$

$$\tau_{ij} = P\cdot\delta_{ij} + \mathcal{D}\,v_i v_j - T\eta_{ijkl}\,\partial_k\left(\frac{v_l}{T}\right)$$

Navier-Stokes:

$$\partial_t\rho + \partial_i J_i = 0 \qquad \partial_t\varepsilon + \partial_i\mathcal{E}_i = 0 \qquad \partial_t g_j + \partial_i\tau_{ij} = 0.$$

Our EFT gives us 2nd law of thermodynamics:

$$\partial_t s + \partial_i S^i \geq 0. \qquad \text{i.e. } \frac{d}{dt}\int d^dx\, s \geq 0.$$

(if we ignore noise).

To derive ... abstract away:

$$\pi^\alpha = \begin{pmatrix} \pi_\varepsilon \\ \pi_\rho \\ \pi_i \end{pmatrix} \qquad \rho^\alpha = \begin{pmatrix} \varepsilon \\ \rho \\ g_i \end{pmatrix} \qquad R^\alpha_j = \begin{pmatrix} (\varepsilon+P)v_j \\ \rho v_j \\ P\delta_{ij} + Dv_iv_j \end{pmatrix} \qquad Q^{\alpha\beta} = \left(\begin{array}{c|c} A & \\ \hline & \eta \end{array}\right) T$$

$\downarrow$

$$\mathcal{L}_{MSR} = \pi^\alpha \partial_t \rho^\alpha - \partial_i \pi^\alpha \left(R^\alpha_i - i Q^{\alpha\beta} \partial_i (\pi^\beta - i\mu^\beta) \right)$$

Noise-free/avg EOM:

$$0 = \partial_t \rho^\alpha + \partial_i \left(R^\alpha_i - Q^{\alpha\beta} \partial_i \mu^\beta \right)$$

Entropy production:

$$\partial_t s = -\mu^\alpha \partial_t \rho^\alpha = \mu^\alpha \partial_i R^\alpha_i - \mu^\alpha \partial_i \left(Q^{\alpha\beta} \partial_i \mu^\beta \right)$$

$$= \underbrace{\cancel{\partial_i \left(\mu^\alpha R^\alpha_i - Q^{\alpha\beta} \mu^\alpha \partial_i R^\beta \right)}}_{\text{ignore, total div}} - \underbrace{R^\alpha_i \partial_i \mu^\alpha}_{\partial_i \tilde{s}^i \text{ (entropy current)}} + \underbrace{Q^{\alpha\beta} \partial_i \mu^\alpha \partial_i \mu^\beta}_{\geq 0}$$

positive semidef $\rightarrow$ $\sigma, \eta, \zeta, \bar{\kappa} \geq 0$, $\sigma \bar{\kappa} \geq T \alpha^2$

$$\boxed{\partial_t s + \partial_i s^i \geq 0}$$

Historically, (Landau) derived hydro

write down most general EOMs $\partial_t \rho + \partial_i J_i = 0 \ldots$

$J_i = \ldots$

consistency w/ thermo & 2nd law

These things come for free from EFT.