PHYS 7810
Hydrodynamics
Spring 2024

Lecture 9
Fluids with Galilean symmetry

February 13

Recap: Navier-Stokes equations for fluid w/ conserved:

$$\left.\begin{array}{lll} \text{mass / (charge)} & \rho \\ \text{energy} & \varepsilon \\ \text{momentum} & g_i \end{array}\right\} \xrightarrow{\text{conjugate to}} \begin{array}{l} \mu_m/T = \mu_\rho \\ -1/T = \mu_\varepsilon \\ v_i/T = \mu_i \end{array}$$

MSR Lagrangian: w/ (micro) time-reversal symmetry

$$\mathcal{L} = \pi_\rho \partial_t \rho + \pi_\varepsilon \partial_t \varepsilon + \pi_i \partial_t g_i - \partial_i \pi_\rho \cdot \rho v_i - \partial_i \pi_\varepsilon (\varepsilon + P) v_i$$
$$- \partial_i \pi_j \left[P \delta_{ij} + g_i v_j \right] + iT \eta_{ijkl} \partial_i \pi_j \left[\partial_k \pi_l - \partial_k i\mu_l \right]$$
$$+ iT \partial_i (\pi_\varepsilon \ \ \pi_\rho) \mathcal{A} \partial_i \begin{pmatrix} \pi_\varepsilon - i\mu_\varepsilon \\ \pi_\rho - i\mu_\rho \end{pmatrix} + iT B_{ijkl} v_i \left[\partial_j \pi_\varepsilon \, \partial_k (\pi_l - i\mu_l) + \partial_j (\pi_\varepsilon - i\mu_\varepsilon) \partial_k \pi_l \right]$$
$$+ \cdots$$

Non-relativistic liquids & gases:

Galilean symmetry $\rightsquigarrow$ space & time trans. (mom./en. cons.)

- ☆ rotation (ang. mom. cons)
- ☆ boost symmetry:

$$v_i \to v_i + u_i$$

Angular momentum:

$$L_{ij} = \int d^dx\, [x_i g_j - x_j g_i] + \cancel{L^{\text{"spin"}}_{ij}} \rightarrow \text{ignore!}$$

$xg \gg$ intensive

$$\frac{dL_{ij}}{dt} = 0 = \int d^dx\, (x_i \partial_t g_j - x_j \partial_t g_i] \qquad \partial_t g_i + \partial_j \tau_{ji} = 0$$

$$= -\int d^dx\, [x_i \partial_k \tau_{kj} - x_j \partial_k \tau_{ki}]$$

$$\underset{\text{IBP}}{=} \int d^dx\, [\tau_{kj}\delta_{ki} - \tau_{ki}\delta_{kj}] = \int d^dx\, \underbrace{[\tau_{ij} - \tau_{ji}]}$$

$\tau_{ij} = \tau_{ji}$ sym.

compatible so far!

Boost symmetry: Microscopic: $P_i = \sum_{\alpha=1}^{N} p_i^\alpha \mapsto \sum_{\alpha=1}^{N} (p_i^\alpha + m u_i)$

$\uparrow$ mass $\uparrow$ boost vel.

$$\rightarrow P_i + M u_i$$

Continuum limit: $g_i \rightarrow g_i + u_i \cdot \rho \leftarrow$ mass density

$$g_i = \frac{\partial P}{\partial(\frac{v^2}{2})} v_i \longrightarrow (v_i + u_i) \frac{\partial P}{\partial(v^2/2)}\left(\rho, \varepsilon, \frac{(v+u)^2}{2}\right)$$

must be equal! $\rho = \frac{\partial P}{\partial(v^2/2)}$

But also: $\rho = \frac{\partial P}{\partial \mu_{th}}$ combine! $P = P(\varepsilon, \underbrace{\mu_{th} + \frac{v^2}{2}}_{\mu})$

Pressure for Galilean-inv: $P(\varepsilon, \rho)$

[physics same in all ref. frames]

Also: $\underline{g_i \to g_i + \rho u_i}$ implies:

$$0 = \partial_t g_i + \partial_j \tau_{ji} \to \partial_t (g_i + u_i \rho) + \partial_j (\tau_{ji} + u_i J_j)$$

mass current

$$= \underbrace{\cancel{[\partial_t g_i + \partial_j \tau_{ji}]}}_{0} + u_i [\partial_t \rho + \partial_j J_j]$$

Before: $\tau_{ji} = P\delta_{ji} + g_j v_i$

u_i = infinitesimal

$$(g_j + \rho u_j)(v_i + u_i) \approx g_j u_i + \rho u_j v_i$$

Claim: $\boxed{J_j = \rho v_j}$

In MSR Lagrangian: $-\partial_i \pi_\rho J^i$

NO! $\cancel{\partial_i \pi_\rho A \partial_i (\pi_\rho - i\mu_\rho)}$ since $J \subset -A\partial_i \mu_\rho$

By Gibbs-Duhem relation:

$$\varepsilon + P = \underbrace{\mu_{th} \frac{\partial P}{\partial \mu_{th}} + T\frac{\partial P}{\partial T}}_{(\varepsilon+P)|_{v=0}(\rho,\varepsilon)} + \frac{v^2}{2}\frac{\cancel{\partial P}^{\,\rho}}{\partial(\frac{v^2}{2})} \longmapsto \underbrace{(\varepsilon_0 + P)}_{v\text{-ind.}} + \frac{\rho v^2}{2}.$$

Lastly: boost invariance + viscosity etc... $\mu_\ell = \frac{v_\ell}{T}$

$$\mathcal{L}_{MSR} \subset iT\eta_{ijk\ell}\partial_i \pi_j \partial_k (\pi_\ell - i\mu_\ell)$$

boost to $\mu_\ell \to \mu_\ell + \frac{1}{T} u_\ell$

$$\partial_k \mu_\ell \to \partial_k \mu_\ell + \underbrace{u_\ell \partial_k \frac{1}{T}}_{\text{bad!}}$$

Recall: $\mu_\varepsilon = -\frac{1}{T}$, so

$$\partial_k\left(\pi_\ell - i\frac{v_\ell}{T}\right) + v_\ell \partial_k\Big(\pi_\varepsilon + \overbrace{\frac{i}{T}}^{-i\mu_\varepsilon}\Big) = \partial_k \pi_\ell + v_\ell \partial_k \pi_\varepsilon - \underbrace{\frac{i}{T}\partial_k v_\ell}_{\text{boost invariant.}}$$

Replace: viscous term w/

$$iT\eta_{ijk\ell}\left[\partial_i \pi_j + v_j \partial_i \pi_\varepsilon\right]\left[\partial_k \pi_\ell + v_\ell \partial_k \pi_\varepsilon - \frac{i}{T}\partial_k v_\ell\right]$$

T

What are allowed η_{ijkl}? Because $\tau_{ij} = \tau_{ji}$ ...

$$\eta_{ijkl} = \eta_{jikl}$$
$$\eta_{klij} = \eta_{ijkl}$$

Most general: $\eta_{ijkl} = \underbrace{\eta}_{\text{shear visc.}} \left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\right] + \underbrace{\zeta}_{\text{bulk visc.}} \delta_{ij}\delta_{kl}$

Collect constraints $\rightarrow$ Navier–Stokes equations:
(neglect noise)

[d = spatial dims]

mass: $\partial_t \rho + \partial_i(\rho v_i) = 0.$

mom.: $\partial_t(\rho v_i) + \partial_j\left[P_0\delta_{ij} + \underbrace{\rho v_i v_j}_{\text{"convection"}} - \underbrace{\eta}_{\text{shear visc}}\partial_i v_j - \eta\partial_j v_i + (\underbrace{\zeta}_{\text{bulk visc.}} - \tfrac{2\eta}{d})\delta_{ij}\partial_k v_k\right] = 0.$

energy: $\partial_t\left(\varepsilon_0 + \frac{\rho v^2}{2}\right) + \partial_i\left[\left(\varepsilon_0 + P_0 + \frac{\rho v^2}{2}\right)v_i - v_j\left[\eta(\partial_j v_i + \partial_i v_j) + (\zeta - \tfrac{2\eta}{d})\delta_{ij}\partial_k v_k\right] - \underbrace{\kappa}_{\text{thermal conductivity}}\partial_i T\right] = 0$

Often momentum re-written:

$$\rho\underbrace{\left[\partial_t v_i + v_j\partial_j v_i\right]}_{\text{"convective der" } \frac{dv_i}{dt}} + \partial_i P - [\text{visc. terms}] = 0.$$

Linearize around equilibrium! $\longrightarrow$ quasinormal modes

$$\left.\begin{aligned} \rho &= \rho_0 + \delta\rho \\ \varepsilon &= \varepsilon_0 + \delta\varepsilon \\ v_i &= 0 + \delta v_i \end{aligned}\right\} \times e^{i(kx - i\omega t)}$$

N-S:

$$0 = -i\omega\,\delta\rho + ik\rho_0\,\delta v_x$$
$$0 = -i\omega\,\delta\varepsilon + ik(\varepsilon_0 + P_0)\,\delta v_x + \kappa k^2\left(\left.\frac{\partial T}{\partial\rho}\right|_\varepsilon \delta\rho + \left.\frac{\partial T}{\partial\varepsilon}\right|_\rho \delta\varepsilon\right)$$

mom. $\begin{cases} 0 = -i\omega\rho_0\,\delta v_x + ik\left(\dfrac{\partial P}{\partial\rho}\delta\rho + \dfrac{\partial P}{\partial\varepsilon}\delta\varepsilon\right) + \left(\dfrac{2d-2}{d}\eta + \zeta\right)\delta v_x k^2 \\ 0 = -i\omega\rho_0\,\delta\vec{v}_\perp + \eta k^2\,\delta\vec{v}_\perp \end{cases}$

Eigenvals/vectors?

① transverse momentum $\delta\vec{v}_\perp \neq \vec{0}$. and $\omega = -i\underbrace{\frac{\eta}{\rho_0}}_{\sim \text{"kinematic visc."}} k^2$

diffusion!

Everything else:

$$i\omega\begin{pmatrix}\delta\rho\\ \delta\varepsilon\\ \delta v_x\end{pmatrix} = \begin{pmatrix} 0 & 0 & ik\rho_0 \\ \cancel{\kappa k^2 \partial_\rho T} & \cancel{\kappa k^2 \partial_\varepsilon T} & ik(\varepsilon_0+P_0) \\ \frac{\partial_\rho P}{\rho_0} ik & \frac{\partial_\varepsilon P}{\rho_0} ik & \cancel{\left(\frac{2d-2}{d}\eta+\zeta\right)\frac{k^2}{\rho_0}}\end{pmatrix}\begin{pmatrix}\delta\rho\\ \delta\varepsilon\\ \delta v_x\end{pmatrix}$$

Start by neglecting k^2. $\rightsquigarrow$ $\omega(k) = \# k + \underbrace{\cdots}_{}$

hydro is EFT on long wavelengths!

$$\omega\delta\rho = k\rho_0 \delta v_x \qquad \omega\delta\varepsilon = k(\varepsilon_0+P_0)\delta v_x$$

$$\hookrightarrow \quad \omega\delta v_x = k\cdot\frac{k}{\omega}\left[\partial_\rho P + \frac{\varepsilon_0+P_0}{\rho_0}\partial_\varepsilon P\right]\delta v_x$$

$\Rightarrow$ ② sound mode: $\omega = \pm v_s k + \cdots$

↑ real → not dissipative

$$v_s^2 = \partial_\rho P + \boxed{\frac{\varepsilon_0+P_0}{\rho_0}}\partial_\varepsilon P = \partial_\rho P + \left.\left(\frac{\partial\varepsilon}{\partial\rho}\right)\right|_{\rho/s}\partial_\varepsilon P = \partial_\rho P\Big|_{\rho/s}$$

Use thermodynamics! $\quad T\,ds = d\varepsilon - \mu_m\, d\rho$

Hold $\gamma = \frac{s}{\rho}$ independent... $\hookrightarrow T\,d(\rho\gamma)$

$$d\varepsilon = \underbrace{(\mu_m + T\gamma)}_{}d\rho + T\rho\, d\gamma$$

$$= \frac{\mu_m\rho + Ts}{\rho} = \boxed{\frac{\varepsilon+P}{\rho}} \quad \text{at } v_i = 0$$

To calculate decay rate: $\sim k^2$ corrections

$$\omega = \pm v_s k - \frac{i}{2}k^2\left[\frac{\kappa\rho_0}{\varepsilon_0+P_0}\left.\frac{\partial T}{\partial\rho}\right|_{\rho/s} + \frac{\frac{2d-2}{d}\eta+\zeta}{\rho_0}\right] + \cdots$$

③ Had 3×3 matrix $(\delta\varepsilon, \delta\rho, \delta v_x)$.

$\hookrightarrow$ 2 sound modes

What's last eigenmode?

$\hookrightarrow$ "incoherent" diffusion mode:

fluctuations in $\delta\varepsilon, \delta\rho$ w/ $\delta P = 0$ $(\delta v_x \approx 0)$

$$\hookrightarrow \quad \delta\rho = -\frac{1}{v_s^2}\left.\frac{\partial P}{\partial \varepsilon}\right|_\rho \qquad \delta\varepsilon = \frac{1}{v_s^2}\left.\frac{\partial P}{\partial \rho}\right|_\varepsilon$$

$$\hookrightarrow \quad \omega = -iDk^2$$

$$D = \frac{\kappa}{v_s^2}\left[\left.\frac{\partial P}{\partial \rho}\right|_\varepsilon \left.\frac{\partial T}{\partial \varepsilon}\right|_\rho - \left.\frac{\partial P}{\partial \varepsilon}\right|_\rho \left.\frac{\partial T}{\partial \rho}\right|_\varepsilon\right]$$

$$\approx \underbrace{\frac{\kappa}{\left(\frac{\partial \varepsilon}{\partial T}\right)}}_{\frac{\text{therm. cond.}}{\text{specific heat}}} \cdot \underbrace{\frac{\partial P/\partial\rho|_\varepsilon}{v_s^2}}_{\approx 1}$$