PHYS 7810: Hydrodynamics — Spring 2026

Homework 1

Due: January 27 at 11:59 PM. Submit on Canvas.

15 **Problem 1 (Time-reversal symmetry in the Langevin equation):** In Lecture 4, we saw how time-reversal symmetry (T) acts on the MSR Lagrangian

$$L = \pi_i \dot{x}_i + \mathrm{i}\pi_i M_{ij}(\pi_j - \mathrm{i}\mu_j). \tag{1}$$

For simplicity, assume that all x_i are T-even, such that $M_{ij} = M_{ji}$ when the above theory is T-symmetric.

Recall the path integral derivation of the MSR Lagrangian from Lecture 4, where we saw that (1) is equivalent to the Lagrangian

$$L = \pi_i\left(\dot{x}_i + M_{ij}\mu_j - b_{i\alpha}\xi_\alpha\right) + \frac{\mathrm{i}}{2}\xi_\alpha\xi_\alpha \tag{2}$$

where

$$M_{ij} = \frac{1}{2} b_{i\alpha} b_{j\alpha}. \tag{3}$$

Find[1] the transformation of ξ_α under time-reversal symmetry, given the known transformations of x_i and π_i derived in Lecture 4, such that L in (2) transforms identically to (1) under time-reversal. Deduce how time-reversal symmetry acts on a Langevin equation directly. Write a sentence or two summarizing how your answer gives a very explicit resolution to the "arrow of time" puzzle in dissipative systems that we raised in Lecture 1.

15 **Problem 2 (Particle under random forcing):** A non-relativistic particle of mass m undergoing random forcing dynamics obeys the equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{m}, \tag{4a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \alpha \cdot \xi \tag{4b}$$

where ξ is Gaussian white noise as in Lecture 1. Assume the initial conditions $x(0) = p(0) = 0$.

Formally integrate these equations to get a formula for $x(t)$ in terms of integral(s) over ξ. Then explicitly show by evaluating these integrals and using the known value of $\langle \xi(s)\xi(s')\rangle$ (given in Lecture 1) that

$$\left\langle x(t)^2 \right\rangle = \frac{\alpha^2}{3m^2} t^3. \tag{5}$$

Give a simple heuristic explanation for the scaling with t.

[1] *Hint:* The fastest way may just be trial and error.

Problem 3 (Spinning galaxy): Consider an anisotropic rigid body with principal moments of inertia $I_1 < I_2 < I_3$ rotating in free space. In classical Hamiltonian mechanics, we can associate this dynamics with a Hamiltonian system where

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \tag{6}$$

and the Poisson brackets are

$$\{L_1, L_2\} = -L_3, \tag{7a}$$
$$\{L_2, L_3\} = -L_1, \tag{7b}$$
$$\{L_3, L_1\} = -L_2. \tag{7c}$$

10 **A:** Show that the Hamiltonian equations of motion $\dot{L}_I = \{L_I, H\}$ ($I = 1, 2, 3$) reproduce Euler's equations of motion for a freely rotating rigid body. Show also that the total angular momentum

$$L^2 = L_1^2 + L_2^2 + L_3^2 \tag{8}$$

is a conserved quantity.

10 **B:** Now suppose that we wish to model the dissipative dynamics of our rotating body. Assume that the dissipation does *not* relax total angular momentum L^2, but it can relax energy. Write down an MSR Lagrangian for this system: $L = \pi_I \dot{L}_I + \cdots$, assuming $\Phi = \beta H$, and explaining the symmetry you impose to mandate that L^2 is conserved as you go. Deduce the noise-free dissipative equations for the rigid body, and in this limit of neglecting noise, determine the late time state of the rigid body. Explain your answer physically.

Although the model we wrote down is too simple because it neglects the internal dynamics of the object which can change I_I, this is a cartoon model for why disk-like galaxies spin. In the galactic context the "friction" in the model above is the conversion of kinetic energy from collective motion of stars into relative motion.

Problem 4 (Rock paper scissors): Consider a theory with three degrees of freedom r, p and s, and a set of stochastic equations governing their dynamics. The equations (e.g. Fokker-Planck equation) are invariant under cyclic permutations of variables, such as

$$\begin{pmatrix} r \\ p \\ s \end{pmatrix} \to \begin{pmatrix} p \\ s \\ r \end{pmatrix} \tag{9}$$

Furthermore, impose a generalized time-reversal symmetry corresponding to

$$\begin{pmatrix} r(t) \\ p(t) \\ s(t) \end{pmatrix} \to \begin{pmatrix} r(-t) \\ s(-t) \\ p(-t) \end{pmatrix}. \tag{10}$$

15 **A:** Write down the most general expression for Φ consistent with these symmetries. Then, write down the most general MSR Lagrangian with generalized time-reversal symmetry and Gaussian noise.

5 **B:** For any choice of parameters (and Φ), is the MSR theory you found above equivalent to

$$\begin{pmatrix} \dot{r} \\ \dot{p} \\ \dot{s} \end{pmatrix} = -\alpha \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ p \\ s \end{pmatrix} + \text{noise?} \tag{11}$$

The details of noise are not important here; $\alpha > 0$ is a constant. Explain your answer. If this is not possible, how does the MSR formalism forbid such a theory?

Problem 5 (Thermal bottlenecks): Consider a generic stochastic system for a single degree of freedom x, with steady state $\Phi(x) = \beta H(x)$, where $\beta = 1/T$ denotes inverse temperature. As in Lecture 3, write the Fokker-Planck equation as

$$\partial_t P = -\hat{W} P = \partial_x \left(Q(x) \left(\partial_x P + P \partial_x \Phi \right) \right). \tag{12}$$

Suppose that you could solve the eigenvalue problem

$$\hat{W} \phi_n = \lambda_n \phi_n. \tag{13}$$

20 **A:** Explain why the problem necessarily has time-reversal symmetry. Then give a physical argument that all $\lambda_n \geq 0$, and explain why at least one λ_n must be 0. Deduce that if $\lambda_n \neq \lambda_m$,[2]

$$\int \mathrm{d}x \; \mathrm{e}^{\Phi(x)} \phi_n(x) \phi_m(x) = 0. \tag{14}$$

Explain how you can then solve the time-dependent Fokker-Planck equation by expanding functions in the eigenbasis ϕ_n.

10 **B:** Now let us consider a model where $H(x) = H(-x)$ is symmetric and

$$H(0) = \Delta + \min_x H(x). \tag{15}$$

Assume for simplicity that $Q(x) = Q$ is a constant. Show that[3] if the temperature T is sufficiently small (how small is necessary?), there is a non-zero eigenvalue λ_1 of $\hat{W}$ scaling as

$$\lambda_1 \sim \mathrm{e}^{-\Delta/T}. \tag{16}$$

Explain the physical implications.

[2] *Hint:* Review from quantum mechanics why the eigenvectors of a Hermitian operator with distinct eigenvalues are orthogonal, and generalize that argument, using time-reversal symmetry where necessary. In general, you may find reviewing how you solve the time-dependent vs. time-independent Schrödinger equation helpful.

[3] *Hint:* Continue the analogy with quantum mechanics and make a variational principle. What is a clever trial function to put into the variational principle?