# Homework 2

**Due:** February 10 at 11:59 PM. Submit on Canvas.

**Problem 1 (Charge screening):** Consider a theory of charged particles living in $d = 3$ spatial dimensions, that interact via long-range Coulomb interactions in vacuum, and are in thermal equilibrium. The total charge is conserved under the dynamics.

15 **A:** Follow the MSR EFT[1] of Lecture 5 to deduce the long-wavelength dynamics of the charge. Argue that (in Fourier space)

$$\Phi \approx \int \mathrm{d}^d k \frac{|\rho(k)|^2}{2\epsilon_0 k^2} + \cdots . \tag{1}$$

where $\cdots$ denotes subleading terms you can neglect. Deduce the following "hydrodynamic" dispersion relation for the conserved charge:

$$\omega = -\mathrm{i}\frac{\sigma}{\epsilon_0}, \tag{2}$$

where $\sigma$ is the conductivity (introduced in the MSR formalism as in Lecture 5).

10 **B:** It might seem inconsistent that charge is conserved and yet $\omega$ does not vanish as $k \to 0$. To see what is happening, it is instructive to build a "microscopic model" for the dynamics. Consider a spherically symmetric distribution of charge, where

$$q(r) = \int_0^r \mathrm{d}s\ 4\pi s^2 \rho(s) \tag{3}$$

is the total charge contained within radius $r$. Combine Gauss' Law and the conservation law for charge with the approximation $\mathbf{J} = \sigma \mathbf{E}$ to derive a differential equation for $q(r, t)$. Solve it and provide physical intuition: how your answer is consistent with **A**, as well as charge conservation.

15 **Problem 2 (Sound poles):** Consider the following simplified MSR Lagrangian from Lecture 7, describing a one-dimensional fluid with only charge and momentum conservation in $d = 1$ dimension:

$$\mathcal{L} = \pi_\rho \partial_t \rho + \pi_g \partial_t g - a\mu_g \partial_x \pi_\rho - a\mu_\rho \partial_x \pi_g + \mathrm{i}\zeta \partial_x \pi_g \partial_x (\pi_g - \mathrm{i}\mu_g), \tag{4}$$

where the steady state

$$\Phi = \int \mathrm{d}x \left[\frac{\rho^2}{2\chi} + \frac{g^2}{2\mathcal{M}}\right]. \tag{5}$$

Here $a, \chi, \mathcal{M}, \zeta$ are all constant. Follow the method developed in Lecture 6 and calculate the retarded Green's function (up to an overall scale) $\langle g(-k, -\omega)\pi_g(k, \omega)\rangle$. Are the poles in this Green's function consistent with the quasinormal modes you expect?

[1] *Hint:* What is the correct choice of $\Phi$ for a system in thermal equilibrium? If charges interact with Coulomb interactions, how can you most easily calculate their electrostatic potential energy? You will need to normalize the coefficient $\sigma$ with an additional factor of temperature $T$, as was discussed in Lecture 7.

**Problem 3 (Angular momentum conservation):** In Lecture 8 we argued that angular momentum conservation allowed us to assume the stress tensor was symmetric. In this problem we will revisit the nature of angular momentum conservation in hydrodynamics more carefully, using the full power of the MSR formalism.

10 **A:** For simplicity, assume that our fluid has momentum density $g_i$ and conserves the total momentum $P_i$ and angular momentum $L_{ij} = -L_{ji}$ given by[2]

$$P_i = \int \mathrm{d}^d x \; g_i(x), \tag{6a}$$

$$L_{ij} = \int \mathrm{d}^d x \; \left[x_i g_j(x) - x_j g_i(x)\right]. \tag{6b}$$

If $\pi_i$ is the MSR-conjugate field to $g_i$, find all possible invariant building blocks for an MSR Lagrangian consistent with $P_i$ and $L_{ij}$ conservation. What can you conclude from this result about the stress tensor $\tau_{ij}$, which we might define in general by

$$L = \pi_i \partial_t g_i - \partial_i \pi_j \cdot \tau_{ij}. \tag{7}$$

20 **B:** If our fluid is made up out of spinning objects which themselves can carry "spin" angular momentum, you might worry that our hydrodynamic description is incorrect because (6b) only includes the "orbital" angular momentum. Let's explore the implications of also accounting for spin angular momentum. Modify

$$L_{ij} = \int \mathrm{d}^d x \; \left[x_i g_j(x) - x_j g_i(x) + s_{ij}(x)\right]. \tag{8}$$

where $s_{ij} = -s_{ji}$ is the spin angular momentum density. Write down an MSR Lagrangian[3]

$$L = \pi_i \partial_t g_i + \frac{1}{2}\pi_{ij}\partial_t s_{ij} + \cdots \tag{9}$$

including all possible invariant terms under $P_i$ and $L_{ij}$ conservation, and assuming time-reversal symmetry (for which $g_i \to -g_i$ and $s_{ij} \to -s_{ij}$). Analyze the resulting quasinormal modes assuming

$$\Phi = \int \mathrm{d}^d x \left[\frac{a}{2} g_i g_i + \frac{b}{4} s_{ij} s_{ij}\right] \tag{10}$$

for $a, b > 0$ and show that $s_{ij}$ decays at a *finite rate* even at arbitrarily long wavelengths. Conclude that it is not a hydrodynamic mode: only momentum $g_i$ is a hydrodynamic mode. Explain physically why this is happening even though angular momentum is also conserved.

[2]This formalism makes sense in general spatial dimensions. In $d = 3$ we can write angular momentum as a pseudovector by writing $L_i = \frac{1}{2}\epsilon_{ijk} L_{jk}$.

[3]The extra factor of 1/2 accounts for the antisymmetry of $s$ and $\pi$, and avoids double counting.

**Problem 4 (Boundary conditions for biased diffusion):** In Lecture 6 we discussed the hydrodynamic effective field theory for charge conservation in $d = 1$ with PT-symmetry:

$$\partial_t \rho + \partial_x J = 0 \tag{11}$$

where the charge current is

$$J = v\rho - D\partial_x \rho. \tag{12}$$

You can assume that $v > 0$ and $D > 0$ are constants. In this problem, we will consider this system on a finite size domain: $0 \le x \le L$.

10 **A:** Suppose that we put periodic boundary conditions on the box, namely $\rho(x,t) = \rho(x+L,t)$. Find a general solution $\rho(x,t)$ to the hydrodynamic equations in terms of a discrete set of quasinormal modes in this finite domain. What is the late time steady-state? What is the time scale on which the system reaches steady-state?

15 **B:** Suppose that instead the boundary conditions are that $J(x=0) = J(x=L) = 0$ – this corresponds to a box with impenetrable walls at the ends. Re-analyze the problem, finding the quasinormal modes and the steady state.

5 **C:** In the finite box, suppose you are handed initial condition

$$\rho(x,0) = \delta\left(x - \frac{L}{2}\right). \tag{13}$$

Qualitatively describe what happens and in particular estimate the time it takes to reach equilibrium. Argue that as $L \to \infty$ this time scale is long compared to the decay rates of quasinormal modes from **B**. Explain why this apparent discrepancy is possible.