PHYS 7810: Hydrodynamics **Spring 2026**

Homework 3

Due: March 3 at 11:59 PM. Submit on Canvas.

Problem 1 (Nozzle): Consider a compressible inviscid flow of gas through a nozzle (Figure 1), with an x-dependent width $w(x)$. You may ignore energy conservation for simplicity in this problem, and further assume that the flow is time-independent and well-approximated by a solution with x-dependent density $\rho(x)$ and $\mathbf{v} \approx v_x(x)\hat{\mathbf{x}}$.

10 **A:** Show that mass conservation, together with reasonable physical boundary conditions, implies that $\partial_x(\rho v_x w) = 0$. Argue that if $\partial_x w \ll 1$,

$$\rho v_x \partial_x v_x + \partial_x P \approx 0. \tag{1}$$

Explain why (1) implies that momentum is not globally conserved – why is this OK?

10 **B:** Suppose that the equation of state for the gas is

$$P(\rho) = \frac{A}{2}\rho^2, \tag{2}$$

for some constant $A > 0$. Show that the hydrodynamic equations imply that

$$\frac{\partial_x w}{w} = \left(\frac{v_x^2}{v_\mathrm{s}^2} - 1\right)\frac{\partial_x v_x}{v_x} \tag{3}$$

where v_s is the local speed of sound.

15 **C:** Qualitatively describe $\rho(x)$ and $v_x(x)$ for a nozzle-shaped $w(x)$. In particular, let x_0 be the point where the nozzle is smallest. Argue that if and only if $v_x(x_0) = v_\mathrm{s}(x_0)$, a solution exists where $v_x > v_\mathrm{s}$ for $x > x_0$. Argue that if this occurs, there can be a stationary shock wave at some $x_\mathrm{s} > x_0$ where the flow again relaxes to be subsonic. (You do not need to determine the location of the shock.)

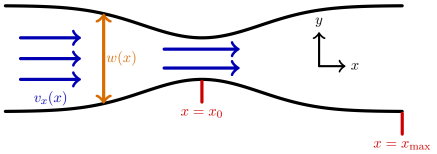

Figure 1: Sketch of flow through a nozzle.

5 **D:** Suppose that the nozzle ends at some $x_{\max}$. Assuming that the pressure has no discontinuity at $x = x_{\max}$ as the gas exits the nozzle, and that the gas outside the nozzle is roughly static, argue that there will be a force acting on the nozzle which is equal in magnitude to

$$F = \left(\rho w v_x^2\right)\Big|_{x=x_{\max}}. \tag{4}$$

Explain why an ideally engineered nozzle which generates the largest possible thrust F would create a shock wave "just past" $x = x_{\max}$.

20 **Problem 2 (Dust in the wind):** Consider a fluid flow in two spatial dimensions in the geometry drawn in Figure 2. The dust particle highlighted in red is semicircular with a radius R and has a mass density $\rho_\mathrm{d} \approx 10^3\ \mathrm{kg/m}^3$, while the fluid has mass density $\rho_\mathrm{f} \approx 1\ \mathrm{kg/m}^3$.

Assume that the dust particle is not moving, while far from the particle, the fluid flows at a uniform velocity $\mathbf{v} = v_0\hat{\mathbf{x}}$ (see Figure 2). Show that you may adapt results from Lecture 10 relatively straightforwardly to obtain the velocity field and pressure. Calculate the net force per unit length acting on the dust particle. Show that the dust particle, sitting in Earth's gravitational field of strength g, will be lifted off of the surface once

$$v_0 > \sqrt{\frac{3\pi}{5}\frac{\rho_\mathrm{d}}{\rho_\mathrm{f}} gR} \tag{5}$$

Deduce v_0 if the dust particle has $R \approx 10^{-5}$ m. Use $g \approx 10\ \mathrm{m/s}^2$. Is your answer reasonable?

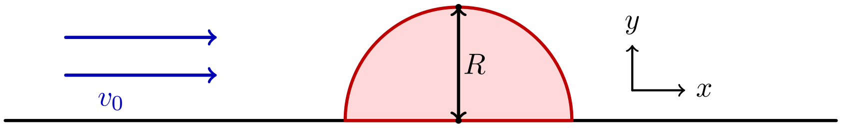

Figure 2: Semicircular dust particle (in red) with radius R. Fluid flows around the dust, with asymptotic velocity v_0.

Problem 3 (Bacterial motion): Many bacteria swim by undulating their body. The goal of this problem is to see that this method is indeed reasonable. To get started, here are some useful numbers: the size of a bacterium is around $\ell \sim 2\times 10^{-6}$ m, the kinematic viscosity of water is $\nu \sim 10^{-6}\ \mathrm{m}^2/\mathrm{s}$, and the bacterium undulates its body at a frequency of about $f \sim 1$ Hz.

10 **A:** Consider the full Navier-Stokes equations for incompressible flow, neglecting energy conservation, as in Lecture 12. Argue that on the time and length scales relevant in the bacterial motion, the Reynolds number is extremely small and $\partial_t \mathbf{v} \ll \nu\nabla^2\mathbf{v}$. Deduce that at any instant in time the fluid motion will look like static creep flow.

15 **B:** A very crude model, working the rest frame of the swimming bacterium, treats the (cell wall of the) bacterium as an infinite plane ($y = 0$), which undulates as follows: the cell wall located at $(x, y = 0)$ at rest is displaced to

$$y \to \delta Y(x,t) = a\cos(kx - \omega t). \tag{6}$$

Treat the amplitude a of oscillation as small. Notice that the flow will be independent of z, so we can treat the flow as two-dimensional. Follow Lecture 12 and look for the stream function $\psi \approx a\psi^{(1)}$ by imposing no-slip boundary conditions, but assuming the boundary is located at $y = 0$: $v_x(y=0) = 0$ and $v_y(y=0) = \partial_t \delta Y$. Then, argue that the solution must be corrected:

$$\psi \approx a\psi^{(1)} + a^2\psi^{(2)} + \cdots \tag{7}$$

where $\psi^{(2)}$ is obtained by demanding that $\psi(x, y = \delta Y(x,t), t)$ obeys the correct boundary conditions, up to order a^2. From the form of ψ, deduce that the velocity u_0 of the bacterium relative to the ambient fluid is

$$u_0 \approx -\frac{a^2 k\omega}{2} \tag{8}$$

Using the numbers provided in the problem, estimate u_0 for a swimming bacterium. The fact that $u_0 \sim a^2$ is a consequence of the difficulty of swimming in a low Reynolds number environment.

Problem 4 (Metabolic rate): In Lecture 12 we discussed how to think of viscous fluid flow through a network of pipes via an analogy to an electrical circuit.

10 **A:** Lecture 12 studied a rectangular pipe in $d = 2$ dimensions. Re-do the derivation for a cylindrical pipe of radius s and length L in $d = 3$ dimensions, and show that the effective resistance is

$$R = \frac{8\eta L}{\pi s^4}. \tag{9}$$

5 **B:** We now reproduce an empirical law in biology, namely that the metabolic rate B at which an animal of mass M consumes energy from food scales as

$$B = \alpha M^{3/4}, \tag{10}$$

for some phenomenological constant α. Consider the circulatory system of the animal as a tree-like network of pipes of radius R_k and length L_k at scale k (see Figure 3). Each pipe at scale k splits into n new pipes at scale $k+1$. Blood (which is incompressible) flows through this network at a velocity which we assume is the same at every scale – show that this implies

$$s_{k+1} = n^{-a} s_k \tag{11}$$

and find the exponent a. Similarly, if each blood vessel at scale k carries blood to all capillaries farther down the tree, which are located within a volume $\sim L^3$, find the exponent b such that

$$L_{k+1} = n^{-b} L_k. \tag{12}$$

We postulate that in all organisms, this tree-like network stops when we reach the scale $k = k_*$ of the capillaries, which are so thin that the blood can deliver nutrients to single cells. We postulate that L_{k_*} and s_{k_*} are organism-independent constants. The metabolic rate $B \sim n^{k_*}$ is proportional to the number of these capillaries. Estimate the total mass M of the organism by the total mass of blood stored in the pipes of the circulatory system. Thus deduce (9).

5 **C:** Now we combine the results of the previous two parts. By analogy to an electrical circuit, determine how the power needed to pump blood through the circulatory system scales with M. How does the pressure difference across the circulatory system scale with M?

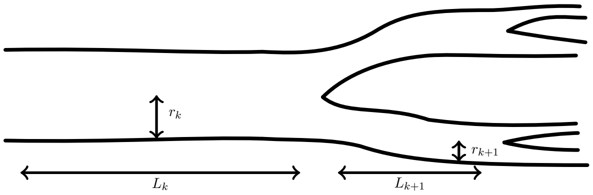

Figure 3: Sketch of a circulatory system with $n = 2$ branchings.