Homework 4

Due: March 24 at 11:59 PM. Submit on Canvas.

15 **Problem 1 (Turbulent drag force on a sphere):** In Lecture 12 we calculated the viscous drag force on a sphere of radius R at small Reynolds number $\mathcal{R} \ll 1$. Assuming that the flow is turbulent when $\mathcal{R} \gg 1$, describe the shape of the wake in turbulent flow around the sphere, using the turbulent viscosity approximation from Lecture 15. Assume $C_\mathrm{D} \sim 1$ and do not worry about O(1) factors in the analysis.

Problem 2 (Flow separation): Consider flow around the wedge depicted in Figure 1. In polar coordinates, flow exists in the region $0 \le \theta \le \pi + \beta$.

10 **A:** Following Lecture 10, study the inviscid and irrotational flow around this object. Argue that close to $r = 0$, the stream function takes the approximate form

$$\psi(r, \theta) = c r^{\pi/(\pi+\beta)} \sin \frac{\pi\theta}{\pi + \beta} \tag{1}$$

for some constant c.

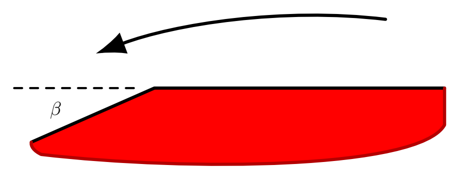

Figure 1: Flow around a wedge; the angle β is illustrated. The red region depicts the "wall" where fluid cannot flow.

15 **B:** At large Reynolds number, we may expect that as in Lecture 13, there is a boundary layer close to $\theta = 0, \pi + \beta$. Picking points along the $\theta = 0$ wall and switching to rectangular coordinates, argue that we should look for a solution to the boundary layer equation (Lecture 13) with

$$\mathbf{v} \approx \tilde{c} x^{-\beta/(\pi+\beta)} \hat{\mathbf{x}} + \cdots \tag{2}$$

outside the boundary layer; here $\tilde{c} = c\pi/(\pi + \beta)$. Show that in terms of similarity variable

$$\xi = \sqrt{\frac{\tilde{c}}{\nu}} \frac{y}{x^m} \tag{3}$$

with $2m = 1 + \beta/(\pi + \beta)$ and ν the kinematic viscosity, you can make a reasonable ansatz for the stream function:

$$\psi = \sqrt{\nu \tilde{c}} x^{1-m} f(\xi). \tag{4}$$

Find the generalization of Blasius' equation for this f.[1] What are the boundary conditions on f?

10 **C:** Solve your nonlinear ODE for f numerically (e.g. with `NDSolve` in `Mathematica`). A simple approach is to integrate the equation from $\xi = 0$ to $\xi \to \infty$ by fixing $f(0)$, $f'(0)$ and "shooting" for the $f''(0)$ that leads to the correct behavior as $\xi \to \infty$. Carry out this prescription and show that a solution fails to exist once $\beta \gtrsim 0.3$ (in radians).

The physical interpretation of the absence of a solution is that there is no laminar boundary layer – the flow will separate away from the wedge. This readily leads to turbulent flow.

[1] *Hint:* You will need to modify the boundary layer equation to deal with the fact that there is an external pressure gradient driving the non-uniform velocity in (2).

Problem 3 (Tsunami): In Lecture 16 we discussed gravity waves. An important application of gravity waves in nature is to the generation of tsunamis, which start off as very long wavelength waves with $k \sim 0.1\ \text{km}^{-1}$, when compared to the typical ocean depth of 4 km.

10 **A:** Re-derive the equations of motion for gravity waves focusing on the limit where $h\partial_x \ll 1$. Neglect surface tension. However, do not drop nonlinear terms. Conclude that[2]

$$\partial_t h + \nabla \cdot (h\mathbf{v}) = 0, \tag{5a}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + g\nabla h = 0. \tag{5b}$$

where ∇ and $\mathbf{v}$ are two-dimensional vectors.

10 **B:** The ocean floor has a naturally variable height $h_0(x)$ which vanishes as we approach the beach. Since $h_0(x)$ varies extremely slowly, you may expect that in the regime where you may approximately linearize (5), you can use a WKB or geometric optics approximation to describe how waves propagate. Show how to derive this approximation in the context of this problem and deduce that the amplitude of a wave moving in the x-direction scales as $h_0^{-1/4}$ as it moves across the ocean.

5 **C:** If a tsunami starts at sea with a wave amplitude of 1 m, estimate how high it could reach as you approach the beach. As part of your estimate, you should deduce where the linear approximation to (5) breaks down; a conservative estimate for the final tsunami height is its height at this location.

Problem 4 (Colorado weather): The Rocky Mountains make weather prediction in Colorado very difficult because wind flow near mountains is extremely unstable. We will explore a crude cartoon for this physics in this problem.

15 **A:** Consider a background flow

$$\mathbf{v} = U(y)\hat{\mathbf{x}}. \tag{6}$$

Write down the inviscid and incompressible Navier-Stokes equations and plug in the linear response ansatz from Lecture 14. Show that

$$k(\omega - kU)v_{y1} = \partial_y \left[v_{y1}\partial_y U + k(\omega - kU)\partial_y v_{y1} \right]. \tag{7}$$

Find the dispersion relation $\omega(k)$ for the ansatz[3]

$$U(y) = U_0\Theta(y). \tag{8}$$

10 **B:** In our cartoon, the mountain exists for $y < 0$ and $x < 0$. Suppose for simplicity that at $x = 0$, we have an approximately homogeneous flow pattern with $v_y(x = 0, y = 0^+) \sim \epsilon(t)$ and $(\partial_x v_y)(x = 0) \approx 0$. For simplicity, assume that the dispersion relation from **A** holds at each ω – you only need to impose the boundary conditions near $y = 0$ as stated above to solve for $v_y(y = 0^+, x, t)$. Describe what happens for $x > 0$ as the fluctuating wind flows over the mountain. Assuming $U_0 \sim 30$ m/s, estimate the length scale over which an extremely slow perturbation would grow ($\dot{\epsilon}/\epsilon \sim (1\ \text{day})^{-1}$).

[2] *Hint:* You may want to proceed a little differently than Lecture 16. First show that $v_z \ll v_{x,y}$. Then you could directly impose conservation of mass and x, y-momentum in a slab of ocean where the height is $h(x, y)$, integrating over z.

[3] *Hint:* You will need to be careful because $\partial_y U = U_0\delta(y)$. This implies that v_{y1} is discontinuous across $y = 0$ with a specific jump, needed in order to make the equations well-behaved.