Homework 5

Due: April 7 at 11:59 PM. Submit on Canvas.

Problem 1 (Rayleigh-Bénard convection): In this problem, you will consider a classic instability to convective heat transfer that arises when a fluid is heated from below. This leads to what is known as Rayleigh-Bénard convection.

Consider a fluid in the Boussinesq approximation from Lecture 18, with density ρ_0 at temperature T_0. The fluid is placed between two infinite plates, one at $z = 0$ and one at $z = h$. Suppose that the acceleration due to gravity is $-g\hat{\mathbf{z}}$. We heat the fluid from below, meaning that for some positive $\Delta T > 0$, we fix boundary conditions such that $T(z = h) = T_0$ while $T(z = 0) = T_0 + \Delta T$.

5 **A:** Find the equilibrium solution to the hydrodynamic equations from Lecture 18, in which $\mathbf{v} = \mathbf{0}$. You should find that the temperature $T_0(z)$ and pressure $P_0(z)$ depend only on z.

5 **B:** We will now show that this solution is unstable. Write

$$P = P_0(z) + P_1(z)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}, \tag{1a}$$

$$T = T_0(z) + T_1(z)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}, \tag{1b}$$

$$\mathbf{v} = \mathbf{v}_1(z)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}. \tag{1c}$$

Show that the hydrodynamic equations reduce to

$$\mathrm{i}kv_{1x} + \partial_z v_{1z} = 0, \tag{2a}$$

$$-\mathrm{i}\omega v_{1x} + \mathrm{i}k\frac{P_1}{\rho_0} - \nu\left(\partial_z^2 - k^2\right)v_{1x} = 0, \tag{2b}$$

$$-\mathrm{i}\omega v_{1z} + \partial_z\frac{P_1}{\rho_0} - \nu\left(\partial_z^2 - k^2\right)v_{1z} - \beta g T_1 = 0, \tag{2c}$$

$$-\mathrm{i}\omega T_1 - \frac{\Delta T}{h}v_{1z} + \chi\left(\partial_z^2 - k^2\right)T_1 = 0. \tag{2d}$$

20 **C:** The physically sensible boundary conditions are that $v_{x1} = v_{z1} = T_1 = 0$ at $z = 0$ and $z = h$. But it is easier to find a solution when we instead take $\partial_z v_{x1} = 0$ at the boundaries. Show that in this case, you may take $T_1(z) = \sin(rz)$ for suitable choices of r. Deduce that

$$\omega = -\frac{\mathrm{i}}{2}\left[(\chi+\nu)q^2 \pm \sqrt{\left((\chi-\nu)q^2\right)^2 + \frac{4\beta g k^2 \Delta T}{hq^2}}\right] \tag{3}$$

where $q^2 = r^2 + k^2$. Find the critical ΔT above which there is an instability of the static equilibrium.

The endpoint of this instability is a dynamical flow with convective "rolls", in which hot fluid rises from the bottom, cools, and then sinks.

20 **Problem 2 (Time-of-flight measurements):** In atomic physics experiments, there is a very elegant way to (approximately) detect the momentum distribution of an interacting gas. Here we illustrate the idea in a simple example. Consider a gas of particles moving in $d = 1$ dimension, which for $t < 0$ are in thermal equilibrium at temperature T in a classical harmonic oscillator potential with single-particle Hamiltonian

$$H = \epsilon(p) + \frac{1}{2} m\omega^2 x^2. \tag{4}$$

We want to determine $\epsilon(p)$ experimentally, assuming that $\epsilon'(p)$ is an invertible function. To do so, suppose that for $t \geq 0$ we release the trap (i.e. set $\omega = 0$). Now the atomic cloud is no longer in thermal equilibrium, and the distribution function $f(x, p, t)$ becomes time-dependent.

Analytically solve the Boltzmann equation, assuming that the collision integral can be neglected, and deduce $f(x, p, t)$ at large t. Explain how the experimentalist can, in prnciple, determine $\epsilon(p)$ based only on the experimentally observable number density

$$n(x, t) = \int_{-\infty}^{\infty} \mathrm{d}p \, f(x, p, t). \tag{5}$$

Problem 3 (Coffee cup): Consider stirring a liquid in a cylindrical cup of radius R and height H at an angular velocity Ω. Suppose that this stirring is done in such a way that there is a steady-state flow where in the *bulk* (away from walls) the flow is approximately rigidly rotating. You do *not* need to do any quantitative calculations – stick to dimensional analysis and ideas from Lecture 17. In this problem, you should work in the rotating reference frame.

10 **A:** Following Lecture 17, argue that there will be an Ekman boundary layer at the bottom of the cup, and that the fluid velocity will have a component with $v_r < 0$. By calculating the inward mass flow in the Ekman boundary layer at different r, deduce that there should be an approximately uniform $v_z > 0$ in the bulk of the cup; estimate how it scales with ν, Ω, R and H.

5 **B:** We might estimate the time scale for the fluid to mix in the container to be the time it takes for a parcel of fluid to rise from the bottom to the top in the flow above. What is this time? Compare this estimate of "mixing time" to that if we only relied on diffusion (viscosity).

10 **C:** By mass conservation, there must be some downward flow near the sides of the cup. To estimate the magnitude of this effect, write (for $r \approx R$, and in a co-rotating frame)

$$\mathbf{v} = v_{\theta 1}(r, z)\hat{\boldsymbol{\theta}} + v_{r1}(r, z)\hat{\mathbf{r}} + v_{z1}(r, z)\hat{\mathbf{z}}. \tag{6}$$

Use the condition $\nabla \cdot \mathbf{v}$ to argue that near $r \approx R$, $v_{r1} \approx \partial_z \psi_1$ and $v_{z1} \approx -\partial_r \psi_1$. Then, neglecting pressure and balancing Coriolis/viscous forces in the boundary layer, and estimating $\partial_z \sim H^{-1}$ and $\partial_r \sim \delta_{\text{side}}^{-1}$, deduce that

$$\delta_{\text{side}} \sim \left(\frac{\nu H^2}{\Omega}\right)^{1/4}. \tag{7}$$

Estimate the velocity downwards near the sides of the cup and sketch the circulatory flow pattern throughout the cup.[1]

[1] This is often called a **secondary flow** since it is induced by the bulk rotational flow together with the no-slip boundary conditions.

25 **Problem 4 (Four-fold rotational symmetry):** Consider a fluid with mass and momentum conservation in $d = 2$. Suppose that the fluid has time-reversal symmetry, but not continuous rotational symmetry nor parity – the only spatial symmetry is under the rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}. \tag{8}$$

Following Lecture 19, write down the most general possible MSR Lagrangian for this fluid, subject to the following simplifying assumptions: you only need to include terms in the constitutive relations for J_i and τ_{ji} that are linear in the velocity v_i, and which contain at most one spatial derivative. Find the quasinormal modes of this fluid (expanding around some finite density, but $v_i = 0$) and compare to a more conventional fluid (e.g. Lecture 8).[2]

[2] *Hint:* You should use `Mathematica` to do the messy symbolic manipulation for you once you need to find the quasinormal mode distribution functions. There is no longer a nice decomposition into longitudinal and transverse modes here!