Homework 6

Due: April 21 at 11:59 PM. Submit on Canvas.

Problem 1 (Transport coefficients of a gas): In this problem we will use kinetic theory to study dissipative coefficients in nonrelativistic monoatomic gases in $d = 3$ spatial dimensions, where

$$\epsilon(\mathbf{p}) = \frac{\mathbf{p}^2}{2m}. \tag{1}$$

10 **A:** We begin by analyzing the collision integral. Argue that rotational symmetry along with momentum and energy conservation imply that the collision rate $R(\mathbf{p}_1\mathbf{p}_2 \to \mathbf{p}_1'\mathbf{p}_2')$ takes the form[1]

$$R(\mathbf{p}_1\mathbf{p}_2 \to \mathbf{p}_1'\mathbf{p}_2') = R(|\mathbf{q}|, \theta)\delta(|\mathbf{q}| - |\mathbf{q}|') \tag{2}$$

where we make a change of coordinates to:

$$\mathbf{P} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} = \frac{\mathbf{p}_1' + \mathbf{p}_2'}{2}, \tag{3a}$$

$$\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2, \tag{3b}$$

$$\mathbf{q}' = \mathbf{p}_1' - \mathbf{p}_2'. \tag{3c}$$

and we define θ as the angle between $\mathbf{q}$ and $\mathbf{q}'$.

10 **B:** Now let us estimate the shear viscosity η and thermal conductivity κ. Explain why:

$$\eta \geq \frac{\langle p_x p_y | p_x p_y\rangle^2}{m^2 \langle p_x p_y | \mathsf{W} | p_x p_y\rangle}, \tag{4a}$$

$$T\kappa \geq \frac{\langle (p^2 - 5mT)p_x | (p^2 - 5mT)p_x\rangle^2}{4m^4 \langle (p^2 - 5mT)p_x | \mathsf{W} | (p^2 - 5mT)p_x\rangle}. \tag{4b}$$

15 **C:** Suppose that these expressions were exact. Evaluate these inner products for a generic $R(\theta)$, using the appropriate inner product for the nonrelativistic gas at temperature T. You should not expect to be able to do the integral over R because you don't know the function. Amazingly, show that the following ratio does not depend on $R(\theta)$:[2]

$$\frac{\eta}{\kappa} = \frac{4m}{15}. \tag{5}$$

10 **D:** Use results from Lecture 21 to show that $c_P = \frac{5}{2}$ for a monoatomic gas (recall the definition of c_P in Lecture 18). Hence deduce an estimate that the Prandtl number of a monoatomic gas is

$$\mathcal{P} = \frac{2}{3}. \tag{6}$$

This can be favorably compared with empirical data for many simple monoatomic gases.

[1] You do not need to justify why there is no $|\mathbf{q}|$ dependence to $R(\theta)$, although this fact is also true!

[2] *Hint:* Use rotational symmetry to explain why: $\langle p^2 p_x | p^2 p_x\rangle = \frac{1}{3}\langle p^2 p_i | p^2 p_i\rangle$ and $\langle p_x p_y | p_x p_y\rangle = \frac{1}{10}\langle p_i p_j - \frac{1}{3}p^2\delta_{ij} | p_i p_j - \frac{1}{3}p^2\delta_{ij}\rangle$. A similar result holds when W is sandwiched in between. By averaging over indices in this way, you should be able to reduce all integrals to an unknown integral over θ and $|\mathbf{q}|$.

20 **Problem 2 (Phonon gas):** Consider a gas of weakly-interacting acoustic phonons (quantized lattice vibrations) in a solid. If the speed of sound in the solid is c, then the energy of a single phonon is

$$\epsilon(\mathbf{p}) = c|\mathbf{p}|. \tag{7}$$

Note that the number of phonons is *not* a conserved quantity; however, we assume that the energy and momentum of the phonon gas are exactly conserved quantities. Because phonons are bosonic excitations, the equilibrium distribution of phonons is given by

$$f_{\mathrm{eq}}(\mathbf{p}) = \frac{1}{\mathrm{e}^{\beta(\epsilon(\mathbf{p})-\mathbf{v}\cdot\mathbf{p})}-1}. \tag{8}$$

Here β is the inverse temperature of the phonon gas, while $\mathbf{v}$ denotes the fluid velocity, as in Lecture 21.

Following Lecture 21, assume that the collisions are extremely fast, and derive the form of the ideal nonlinear hydrodynamic equations, governing energy and momentum conservation. Work in a generic number of spatial dimensions d.[3]

Problem 3 (Ferromagnetism): Consider a ferromagnet: an ordered phase of matter in which a global SO(3) symmetry (such as spin/flavor rotation symmetry) is spontaneously broken to SO(2). As in Lecture 24, we could then try to write down an MSR Lagrangian for a "superfluid" EFT incorporating the Goldstone bosons of the spontaneously broken symmetry. The philosophy is as follows: there are three conserved charge densities ρ_a ($a = 1, 2, 3$) associated with the SO(3) symmetry. The Poisson brackets between these conserved charge densities are:

$$\{\rho_a, \rho_b\} = \epsilon_{abc}\rho_c. \tag{9}$$

In a ferromagnet, we have in equilibrium that $\rho_3 \neq 0$ while $\rho_1 = \rho_2 = 0$.[4] Because of this fact, we say that ρ_1 and ρ_2 are spontaneously broken symmetries – they do not commute with $\rho_3 \neq 0$. We then expect two Goldstone bosons ϕ_A ($A = 1, 2$) associated to these degrees of freedom, with Poisson brackets $\{\rho_a, \phi_A\} = \delta_{Aa}$.

15 **A:** Follow Lecture 24 and write down a general MSR Lagrangian for ρ_a and ϕ_A, together with their conjugate noise variables. Include only the simplest possible dissipative motifs, e.g. $\mathrm{i}\sigma_3\partial_i\pi_3\partial_i(\pi_3-\mathrm{i}\mu_3)$, and assume that $\Phi(\rho_a, \phi_A)$ is a simple quadratic function of ρ_a and ϕ_A. Be careful to include all Poisson bracket terms in the MSR Lagrangian!

20 **B:** Find the quasinormal modes of the ferromagnet, expanding around $\rho_a = (0, 0, \bar{\rho})$ and $\phi_A = 0$. You should find that although there were two Goldstone bosons, there is only "one" propagating degree of freedom with a quadratic dispersion $\omega \sim \pm k^2$. Explain why the Goldstone boson in a ferromagnet behaves qualitatively differently to the one in the U(1) superfluid, discussed in Lecture 24.

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[3] *Hint:* Write $v_i = (v, 0, \ldots, 0)$ and $v_i p_i = v|p|\cos\theta$. Then you may use the fact that rotational symmetry in the remaining dimensions leads to $\int \mathrm{d}^d p \to \int_0^\infty \mathrm{d}p \int_0^\pi \mathrm{d}\theta \times 2\pi^{(d-1)/2}\Gamma(\frac{d-1}{2})^{-1} p^{d-1}\sin^{d-2}\theta$. Feel free to use `Mathematica` to do any of the integrals over p and θ, and don't worry about formulas for O(1) constants.

[4] The non-zero component of the density can be chosen this way without loss of generality by a global rotation.