PHYS 7810
Hydrodynamics
Spring 2026

Lecture 1
Stochastic differential equations

January 8

Welcome! Begin w/ a few brief administrative remarks:

- Class in-person only, no recordings.
- If traveling, Zoom lecture if possible; otherwise pre-recorded.
- All documents (incl. lecture notes) on Canvas homepage
- Grade is based on 6? HWs.
- You can turn in a HW 2 days late OR drop it twice.
- DO NOT copy AI and turn it in. This is cheating.
- For lawyerly commentary please see course syllabus

This is an advanced class covering modern approach to hydrodynamics. There are not good books or reviews on the subject in general that go into too much detail. The lecture notes will be posted frequently and contain the calculations from lecture, but may miss important commentary. So I encourage you to attend in-person.

Hydrodynamics = dissipative effective theory describing how many-body system approaches equilibrium.

effective theory = phenomenological description based on symmetry, independent of microscopics, valid on large scales

dissipative = "irreversibility"/arrow of time. (2^{nd} law)

How should we systematically combine these ideas?

Example: 1d particle with translation symmetry $x \to x + \varepsilon$.

If the theory was dissipationless we would write down a Lagrangian. Assuming we have time-translation symmetry, the invariant Lagrangian is

$$L = \frac{1}{2} m \dot{x}^2 + \underbrace{\cdots}_{\text{higher-order}} \xrightarrow{\text{EOM}} m\ddot{x} = 0.$$

"everything" consistent w/ symmetry

add dissipation by hand?

$$m\ddot{x} + \gamma\dot{x} + \cdots = 0$$

friction force

Solve these equations to find

$$\dot{x}(t) = \dot{x}(0)\, e^{-\gamma t/m} \quad \Rightarrow \quad \gamma/m > 0$$

so particle doesn't accelerate forever.

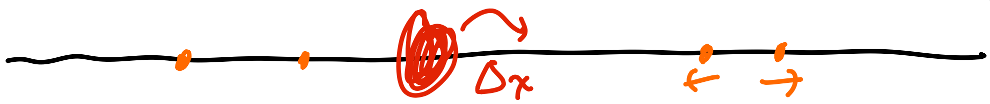
Issues:

1) positivity conditions are awkward. We'd like to fix them before we write down the effective theory.

2) invariant Lagrangian > covariant EOM

3) time-reversal symmetry lost? Maybe natural due to dissipation. On the other hand does microscopic T have zero consequences? We'll see that all of these issues solved by adding noise to our description of a dissipative system! How to include noise??

To understand how to couple noise to a differential equation let's begin by thinking about a toy cartoon microscopic model (not eff. th) w/ noise-dominated dynamics: the random walk.

Example: 

big particle hitting small particles $\Rightarrow$ Brownian motion.

Each collision: $\Delta x_n = \pm \ell$ (each 50% probability)

occur in $\Delta t = \tau$.

Write: $v_n = \frac{\Delta x_n}{\Delta t} = \pm \frac{\ell}{\tau}$. Then

$$x(n\tau) - x(0) = \sum_{j=1}^{n} \Delta x_j = \tau \sum_{j=1}^{n} v_j$$

on average: $\langle x(n\tau) - x(0) \rangle = \tau \sum_{j=1}^{n} \langle v_j \rangle = 0.$

$\langle v_j \rangle = \frac{1}{2}\left(+\frac{\ell}{\tau}\right) + \frac{1}{2}\left(-\frac{\ell}{\tau}\right) = 0$

But: $$\langle [x(n\tau) - x(0)]^2 \rangle = \tau^2 \sum_{j,j'=1}^{n} \langle v_j v_{j'} \rangle$$

$$= \tau^2 \sum_{j=1}^{n} \langle v_j^2 \rangle + \tau^2 \sum_{j \neq j'} \langle v_j v_{j'} \rangle$$

$\langle v_j v_{j'} \rangle = \langle v_j \rangle \langle v_{j'} \rangle = 0$, noise <u>independent</u>

$$= \tau^2 \cdot n \cdot \left(\frac{\ell}{\tau}\right)^2 = n\ell^2 = \frac{\ell^2}{\tau} \cdot t$$

$\ell^2/\tau = 2D$, where D = diffusion constant.

We'd like to describe this by a differential equation / effective theory on long time scales. Clearly we can't use a usual ODE, randomness was essential here!

Goal: stochastic differential equation (Langevin equation)

↳ formal? $\frac{dx}{dt} = \hat{\xi}(t)$? But what is $\hat{\xi}(t)$?

Set $\langle x(0)\rangle = 0$. We need:

$\langle x(t)\rangle = 0$. $\langle \cdots \rangle$ means "average over $\hat{\xi}$" ?

Formally: $x(t) = \int_0^t ds \left(\frac{dx}{dt}\right)(s) = \int_0^t ds\, \hat{\xi}(s)$

$$\langle x(t)\rangle = \int_0^t ds \langle \hat{\xi}(s)\rangle = 0 \rightarrow \langle \hat{\xi}\rangle = 0?$$

b/c expectation value is linear.

$$\langle x(t)^2\rangle \overset{?}{=} 2Dt$$

$$= \int_0^t ds_1 \int_0^t ds_2 \langle \hat{\xi}(s_1)\hat{\xi}(s_2)\rangle$$

local in time

↳ Guess: $= 2D\,\delta(s_1 - s_2)$

$$= \int_0^t ds_1 \int_0^t ds_2 \cdot 2D\,\delta(s_1 - s_2) = \int_0^t ds_1\, 2D = 2Dt$$

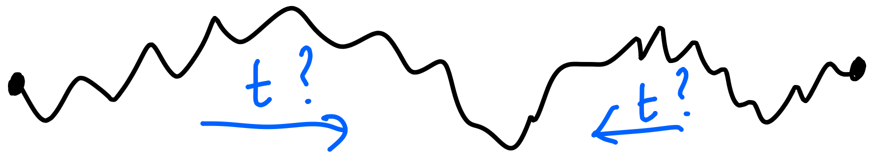
Conventional to write: $\frac{dx}{dt} = \sqrt{2D}\,\xi(t)$

Gaussian white noise: $\langle \xi\rangle = 0$, $\langle \xi(t)\xi(s)\rangle = \delta(t-s)$.

Units of $[\xi] = \frac{1}{\sqrt{[\text{time}]}}$. $\Rightarrow$ noise dominant on small time scales!

With this mathematical abstraction we can start to formally think about systematically modeling noisy dynamics.

Note: Brownian motion has time-reversal symmetry



In lectures 2–3 we'll see how this resolves the puzzle about time-reversal w/ damping / friction!

What about a more generic stochastic equation?

Example: $\dfrac{dx}{dt} = a(x,t) + b(x,t)\,\xi(t)$

$\hookrightarrow$ $dx = a(x,t)\,dt + b(x,t)\,d\xi$

Claim: this equation is ambiguous!

To see this explicitly let's follow trajectory over very short time..

Given $x(0)$, what is $x(dt)$? (On average?)

Itô perspective: $\langle dx \rangle = \langle a(x,t)\,dt \rangle + \langle b(x,t)\,d\xi \rangle$

Plug in $x \to x(0)$

$= a(x(0),0)\,dt + \cancel{b(x(0),0)\langle d\xi \rangle}$

Stratonovich perspective: Plug-in: $x \to \frac{1}{2}x(0) + \frac{1}{2}x(dt)$

$= x(0) + \frac{1}{2}dx$:

$$dx = a(x(0))\,dt + b(x(0))\,d\xi + \tfrac{1}{2}\partial_x a \cdot dx\,dt + \tfrac{1}{2}\partial_x b \cdot dx\,d\xi$$

$$\hookrightarrow \quad dx = \frac{b\,d\xi + a\,dt}{1 - \frac{1}{2}\partial_x a\,dt - \frac{1}{2}\partial_x b\,d\xi}$$

Since $d\xi = \xi\, dt$, $[d\xi] = \sqrt{[\text{time}]}$, so $d\xi^2 \sim dt$?

$$\langle d\xi^2 \rangle = \left\langle \int_0^{dt} ds_1\, \xi(s_1) \int_0^{dt} ds_2\, \xi(s_2) \right\rangle = dt.$$

Thus $\langle dx \rangle = a\, dt + \cancel{b \langle d\xi \rangle} + \frac{1}{2} b \partial_x b \langle d\xi^2 \rangle + \underbrace{\cdots}$

higher-order terms $d\xi dt$, etc. can be ignored as $d\xi \to 0$

$$\langle dx \rangle = \left(a + \frac{1}{2} b \partial_x b\right) dt.$$

We see that Itō & Stratonovich formulations are inequivalent. To fix this problem and find an unambiguous formulation of stochastic eqs (important for EFT) it will be helpful to take a different perspective, which we do in lecture 2.