PHYS 7810
Hydrodynamics
Spring 2026

Lecture 10
Incompressible inviscid flows

February 10

Today we'll begin discussing the simplifications in Navier–Stokes eqns that govern everyday flows of fluids, especially liquids.

Assume:
- energy conservation negligible.
- no dissipation/viscosity (today)
- $\rho \approx$ constant $\longrightarrow$ incompressible flow

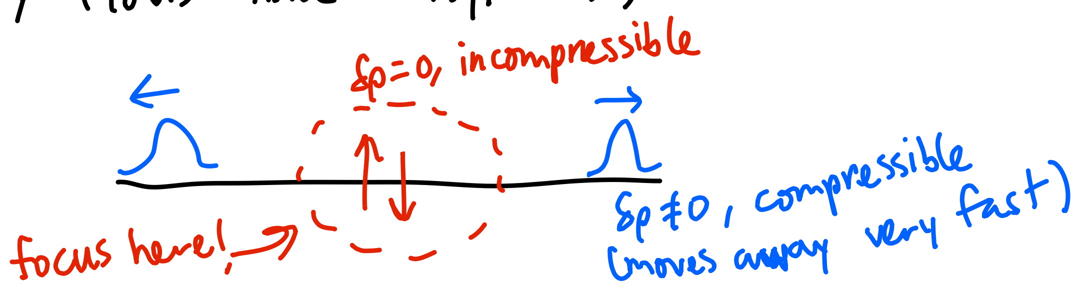
Navier-Stokes: $\cancel{\partial_t \rho} + \partial_i(\rho v_i) = 0 \longrightarrow \partial_i v_i = 0$

$$\rho[\partial_t v_i + v_j \partial_j v_i] + \partial_i P = 0$$

Why is incompressible OK? Fluids have $\underbrace{\text{sound waves}}_{\text{compressible dynamics: } \omega = \pm v_s k}$:

For ambient conditions: liquid (water) : $v_s \sim 1500$ m/s
gas (air) : $v_s \sim 330$ m/s

Most everyday flows have $|v_i| \ll v_s$, so:



let $\rho = \rho_0 + \frac{1}{v_s^2}\chi$ ← compressible perturbation.

ρ_0 ↖ const.

$$\rho_0 \partial_i v_i + \frac{\rho_0}{v_s^2}\cancel{\left[\partial_t \chi + \partial_i(\chi v_i)\right]} = 0$$

$$\partial_t v_i + v_j \partial_j v_i + \frac{\partial_i P}{\rho} = 0$$

} Keep leading order terms in v_s

$$\frac{\partial_i P}{\rho} = \frac{\partial P}{\partial \rho}\frac{\partial_i \rho}{\rho} = v_s^2\left(\frac{1}{\rho_0} - \cancel{\frac{\chi}{v_s^2}\frac{1}{\rho_0^2}} + \dots\right)\frac{\partial_i \chi}{v_s^2} = \frac{\partial_j \chi}{\rho_0}$$

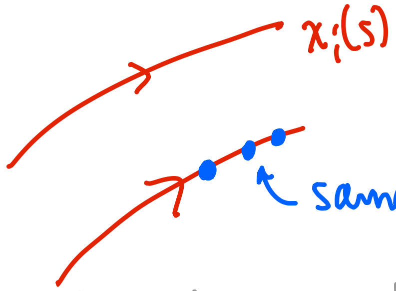
Hence: $\partial_i v_i = 0$ and $\partial_t v_i + v_j \partial_j v_i + \underbrace{\frac{\partial_i \chi}{\rho_0}}_{\text{usually write as } \frac{\partial_i P}{\rho_0}.} = 0$

It's not that ρ = constant holds exactly in incompressible flow, but that tiny $\nabla\rho$ can make huge ∇P...

If $\partial_t = 0$ (static flow):

$$v_i\left[v_j \partial_j v_i + \frac{\partial_i P}{\rho_0}\right] = 0 = v_j \partial_j \underbrace{\left[\frac{v^2}{2} + \frac{P}{\rho_0}\right]}$$

= constant along streamlines: $\frac{dx_i(s)}{ds} = v_i(x_i(s))$



← same $\frac{v^2}{2} + \frac{P}{\rho_0}$: Bernoulli's equation

We'll find this equation useful later to help us calculate forces on surfaces suspended in fluid flows.

Today: assume irrotational flow: $\nabla \times \vec{v} = \vec{0}$

$\varepsilon_{ijk}\partial_j v_k = 0$ $(d=3)$

$v_i = \partial_i \Phi$ ← $\partial_i v_j - \partial_j v_i = 0$ (general d)

↖ velocity potential

Since $\partial_i v_i = 0 \rightsquigarrow \nabla^2 \Phi = 0$ (Laplace's equation)

We can then bring what we know from E&M about the solns to Laplace's equation. But in fluids a particularly useful technique is conformal mapping, which is often not taught in E&M...

Work in $d=2$: $\nabla^2 \Phi = 0 \Rightarrow \Phi = \mathrm{Re}\left[\underbrace{w(z)}\right]$, $z = x+iy$

real part of

holomorphic function: no singularity ($\log z$, $\sqrt{z}$ ...)

Proof: $(\partial_x^2 + \partial_y^2)\Phi = \underbrace{(\partial_x - i\partial_y)}_{2\partial/\partial z}\underbrace{(\partial_x + i\partial_y)}_{2\partial/\partial \bar z}\Phi$ where $\bar z = x - iy$

$$= \mathrm{Re}\left[4\frac{\partial}{\partial z}\cancel{\frac{\partial}{\partial \bar z} w(z)}\right] = 0$$

Moreover, $\Phi(x,y) + i\underbrace{\Psi(x,y)} = w(x,y)$

stream function: constant along streamlines

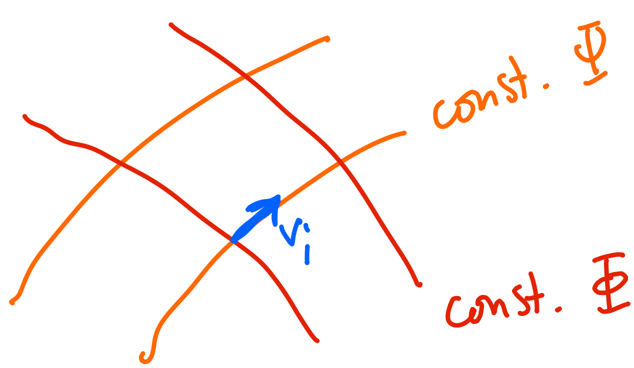
$$2\frac{\partial}{\partial \bar z} w = 0 = \left[\partial_x \Phi - \partial_y \Psi\right] + i\left[\partial_y \Phi + \partial_x \Psi\right]$$
$$= \left[v_x - \partial_y \Psi\right] + i\left[v_y + \partial_x \Psi\right]$$

$$v_x = \partial_y \Psi \qquad v_y = -\partial_x \Psi$$

so $v_x \partial_x \Psi + v_y \partial_y \Psi = 0$ and $\Psi = \text{const.}$ on streamlines

Moreover, $\frac{dw}{dz} = \frac{1}{2}(\partial_x - i\partial_y)(\Phi + i\Psi) = v_x - iv_y$.

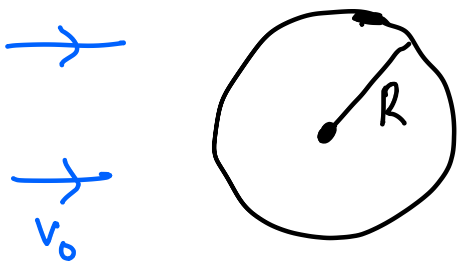
So for $d=2$, irrotational + incompressible flow, elegant picture:



with holomorphic $w(z) = \Phi + i\Psi$

This is an extremely efficient way to do computations, as we'll now show.

Example: flow around disk



flow uniform, fixed velocity $v_0\hat{x}$ far from disk.

Boundary conditions? $\vec{v}\cdot\hat{n}=0$ (mass can't penetrate...)

↳ as $r\to\infty$: $w(z)\to v_0 z$ (v_0 real)

at $r=R$: $\Psi(|z|=R) = \text{const} = 0$

Often one proceeds by making an inspired guess, or looking up someone else's good guess!

$$w(z) = v_0 z + \boxed{\frac{c}{z}} \rightarrow v_0 r e^{i\theta} + \frac{c}{re^{i\theta}}$$

Singularity at $z=0$ is fine since fluid doesn't flow here!

$$\Psi = \text{Im}(w) = \underbrace{v_0 r\sin\theta - \frac{c}{r}\sin\theta}_{=0 \text{ at } r=R \text{ if } c=v_0R^2}$$

$$v_x - iv_y = \frac{dw}{dz} = v_0 - \frac{v_0R^2}{z^2} = v_0\left[1-\frac{R^2}{(x^2+y^2)^2}(x^2-y^2-2ixy)\right]$$

Now let's show that in general we can calculate the net force on an object using Bernoulli's equation:

Newton's 3rd Law: $$F_i = -\oint_{\text{bdy}} dA_j\, \tau_{ji} = -\oint_{\text{bdy}} dA_j\left[P\delta_{ij} + \rho_0 v_i v_j\right]$$

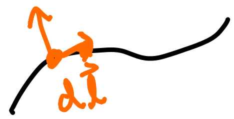
$$= -\oint_{\text{bdy}} dA_i\, P \quad \text{since } dA_jv_j=0 \text{ by BCs}$$

Apply $P+\rho_0\frac{v^2}{2}=\text{const.}$ to streamline at surface of object!

$$F_i = -\oint dA_i \left[\text{const.} - \rho_0 \frac{v^2}{2}\right] = \oint dA_i \, \frac{\rho_0}{2} \left|\frac{dw}{dz}\right|^2$$

normal vector

$dA_i = \hat{n}_i |d\vec{l}|$



and $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

$\hookrightarrow d\vec{A} = dx\,\hat{y} - dy\,\hat{x}$

so $$F_x - iF_y = \oint \left[-i\,dx - dy\right] \frac{\rho_0}{2} (v_x + iv_y)(v_x - iv_y)$$

Now: since $\vec{v}\cdot d\vec{A} = 0 \rightarrow -v_x\,dy + v_y\,dx = 0$

$$(i\,dx + dy)(v_x + iv_y) = i\left[v_x dx + v_y dy\right] \pm \left[v_x dy - v_y dx\right]$$
$$= -(i\,dx - dy)(v_x - iv_y) = -i\,dz\,(v_x - iv_y).$$

$$\underbrace{F_x - iF_y = i\frac{\rho_0}{2} \oint dz \left(\frac{dw}{dz}\right)^2}_{\text{Blasius' Theorem}}$$

which we can analyze using tools from complex analysis!

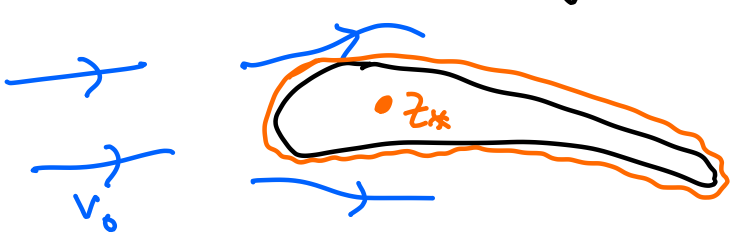
Returning to our example we see that:

$$F_x - iF_y = \frac{i\rho_0}{2} \oint_{|z|=R} dz \left(v_0 - \frac{v_0 R^2}{z^2}\right)^2 = 0$$

no simple pole $1/z$ enclosed!

So there's no net force on our disk! This surprising fact will no longer hold when accounting for viscosity (Lecture 12).

To design airplane wing, we want $F_x - iF_y \neq 0$! So:



engineer stream function so simple pole at $z = z_*$?

Try: $$w(z) = v_0 z - \frac{i\Gamma}{2\pi}\log z + \cdots \quad \text{as } z \to \infty.$$
$$\frac{dw}{dz} = v_0 - \frac{i\Gamma}{2\pi z} + \cdots$$

Evaluate Blasius' Thm on contour very far from wing:

$$F_x - iF_y = \frac{i\rho_0}{2} \oint_{\text{far}} dz \left(v_0 - \frac{i\Gamma}{2\pi z} + \cdots \right)^2 = \frac{i\rho_0}{2} \oint dz \left(v_0^2 - \frac{2i\Gamma v_0}{2\pi z} + \cdots \right)$$

$$= \underbrace{i\,\rho_0 \Gamma v_0}$$ by Cauchy's Thm.

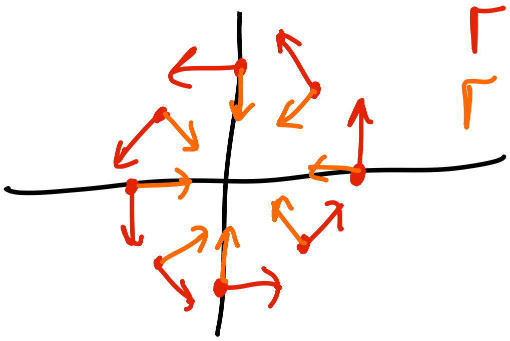
Kutta-Zhukovsky Thm: lift force $F_y = -\rho_0 v_0 \cdot \mathrm{Re}(\Gamma)$

What flow corresponds to $\Gamma \neq 0$? let's see!

Try: $w(z) = -\frac{i\Gamma}{2\pi} \log z$

$$v_x - iv_y = \frac{dw}{dz} = \frac{i\Gamma}{2\pi z} = \frac{i\Gamma}{2\pi} \frac{x - iy}{x^2 + y^2}$$

$$(v_x, v_y) = \frac{\Gamma}{2\pi(x^2+y^2)} (y, -x) :$$



Γ real: vortex

Γ imaginary: mass source or sink, which is unphysical!