PHYS 7810
Hydrodynamics
Spring 2026

Lecture 11
Vortex dynamics

February 12

Vorticity: $\omega_{ij} = \partial_i v_j - \partial_j v_i$

$\hookrightarrow d=3$; $\omega_i = \varepsilon_{ijk}\partial_j v_k \qquad (\vec{\omega} = \nabla \times \vec{v})$

Again assuming incompressible flow: "$\rho =$ const." and $\nabla \cdot \vec{v} = 0$.

Navier-Stokes: $\partial_t v_i + v_j \partial_j v_i + \frac{1}{\rho}\partial_i P = \frac{\eta}{\rho}\partial_j \partial_j v_i$

$\frac{\eta}{\rho} = \nu$, kinematic viscosity

useful numbers for air: $\rho \sim 1\ \mathrm{kg}/\mathrm{m}^3$ & $\nu \sim 10^{-5}\ \mathrm{m}^2/\mathrm{s}$

let's rewrite Navier-Stokes to directly solve for ω_i:

$$\partial_t \omega_i + \varepsilon_{ijk}\partial_j \left[v_\ell \partial_\ell v_k + \cancel{\frac{1}{\rho}\partial_k P} \right] = \nu \partial_\ell \partial_\ell \omega_i$$

$$\hookrightarrow = v_\ell \partial_\ell \omega_i + \underbrace{(\partial_\ell v_k)\varepsilon_{ijk}\partial_j v_\ell}$$

$$= (\partial_k v_\ell + \varepsilon_{\ell k m}\omega_m)\varepsilon_{ijk}\partial_j v_\ell$$

$$= \cancel{\varepsilon_{ijk}(\partial_j v_\ell)(\partial_k v_\ell)} + \omega_m(\delta_{im}\delta_{j\ell} - \delta_{jm}\delta_{i\ell})\partial_j v_\ell$$

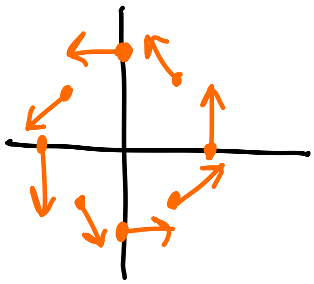
$$= -\omega_m \partial_m v_i$$

Therefore: $\partial_t \omega_i + v_j \partial_j \omega_i = \omega_j \partial_j v_i + \nu \partial_j \partial_j \omega_i$

In 2d flow: $\partial_z = 0$ and $v_z = 0$:

$$\underbrace{\partial_t \omega_z + \vec{v}\cdot\nabla\omega_z}_{\text{vortex is "convected"}} = \underbrace{\nu\nabla^2\omega_z}_{\text{vorticity diffuses}}$$

Example 1: isolated point vortex (Lecture 10)



$$v_x = \frac{\Gamma}{2\pi}\frac{-y}{x^2+y^2} \qquad v_y = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$$

$\hookrightarrow$ $\vec{v} = \frac{\Gamma}{2\pi r}\hat{\theta}$ in polar coords.

Claim: $\omega_z = \partial_x v_y - \partial_y v_x = \Gamma\delta(x)\delta(y)$

because $\underset{\text{circle, radius } R}{\oint} \vec{v}\cdot d\vec{s} = 2\pi R\cdot\frac{\Gamma}{2\pi R} = \Gamma$, ind. of R

This is an exact solution without viscosity because

$$\partial_t\omega_z + \vec{v}\cdot\nabla\omega_z = \partial_t\omega_z + \frac{\Gamma}{2\pi R}\partial_\theta\omega_z = 0$$

Add viscosity: $\partial_t\omega_z + \underbrace{\cancel{\vec{v}\cdot\nabla\omega_z}}_{\text{rotational symmetry}} = \nu\nabla^2\omega_z$

From lecture 2's solution to diffusion equation: $\omega_z = \frac{e^{-r^2/4\nu t}}{4\pi\nu t}\Gamma$

Since $\vec{v} = v_\theta(r)\hat{\theta}$ and $\omega_z = \nabla\times\vec{v} = \frac{1}{r}\partial_r(rv_\theta)$:

$$r v_\theta(r)\Big|_{r_0}^{\infty} = \int_{r_0}^{\infty} dr\cdot r\,\omega_z(r)$$

$$\frac{\Gamma}{2\pi} - r_0 v_\theta(r_0) = \int_{r_0}^{\infty} dr\,\frac{\Gamma}{4\pi\nu t}r\,e^{-r^2/4\nu t} = \frac{\Gamma}{2\pi}e^{-r^2/4\nu t}$$

large scale motion won't decay immediately!

Hence: $v_\theta(r) = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/4\nu t}\right)$

We can also use Navier-Stokes to calculate pressure:

By rotational symmetry, $\partial_\theta P = 0$, so:

$$\cancel{\partial_t v_r} + (\vec{v}\cdot\nabla\vec{v})_r + \frac{1}{\rho}\partial_r P = \nu(\nabla^2 v)_r$$

$$\xrightarrow{v_r = 0} \quad -v_\theta^2/r + \frac{1}{\rho}\partial_r P = \nu\left(-\frac{2}{r^2}\partial_\theta v_\theta\right) = 0$$

$$\hookrightarrow \quad \frac{P_0 - P(r_0)}{\rho} = \int_{r_0}^{\infty} dr\, \frac{v_\theta^2}{r}$$

At $r=0$: $P_{center} = P_0 - \rho\frac{\log 2}{4\nu t}\left(\frac{\Gamma}{2\pi}\right)^2$

Tornado: $R \sim 100$ m across, speed $v_0 \sim 50$ m/s

$\hookrightarrow \frac{\Gamma}{2\pi} \sim v_0 R$ and $4\nu t \sim R^2$:

$P_{center} \sim P_0 - \boxed{\rho v_0^2} \rightarrow$ pressure deficit ~ 2.5 kPa $\sim \frac{1}{40}$ atm

Example 2: multiple point vortices, neglecting dissipation

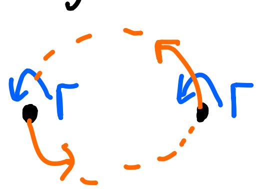
$\hookrightarrow \partial_t \omega_z + (\vec{v}\cdot\nabla)\omega_z = 0$

Each vortex will follow trajectory set by velocity of others

$$\omega_z = \sum_{\alpha=1}^{N} \Gamma_\alpha\, \delta(\vec{r} - \vec{r}_\alpha(t))$$

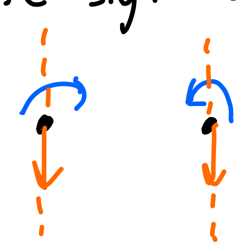
$$\hookrightarrow \frac{d}{dt}\vec{r}_\alpha = \sum_{\beta\neq\alpha} \frac{\Gamma_\beta}{2\pi}\frac{\hat{z}\times(\vec{r}_\alpha - \vec{r}_\beta)}{|\vec{r}_\alpha - \vec{r}_\beta|^2}$$

same-sign vortices:



trace out a circle!

opposite sign vortices:



follow straight line

Amazingly we can show that the point vortex dynamics described above is a Hamiltonian dynamical system!

Consider Poisson bracket: $\{x_\alpha, y_\beta\} = \frac{1}{\Gamma_\alpha}\delta_{\alpha\beta}$.

Hamiltonian $H = -\sum_{\alpha\neq\beta} \frac{\Gamma_\alpha\Gamma_\beta}{2\pi}\log|\vec{r}_\alpha - \vec{r}_\beta|$

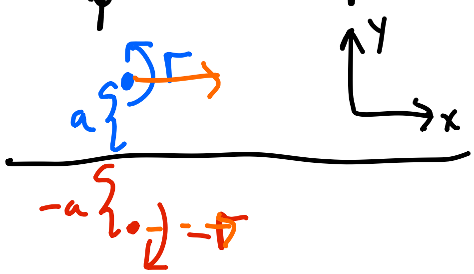
$$S = \int dt\left[\sum_\alpha \Gamma_\alpha y_\alpha \dot{x}_\alpha + \sum_{\alpha\neq\beta}\frac{\Gamma_\alpha\Gamma_\beta}{2\pi}\log|\vec{r}_\alpha - \vec{r}_\beta|\right]$$

A short calculation verifies the construction:

$$\dot{x}_\alpha = \{x_\alpha, H\} = \frac{1}{\Gamma_\alpha}\frac{\partial H}{\partial y_\alpha} = -\sum_{\beta\neq\alpha}\frac{\Gamma_\beta}{2\pi}\frac{y_\alpha - y_\beta}{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}$$

This physics is often used to describe vortices in superfluid thin films, or particles in the quantum Hall regime in solids. Can even be used for modeling 2d turbulence...!

Example 3: point vortex near a wall



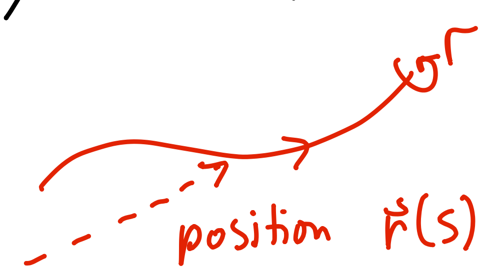
Solve by method of images!

$\hookrightarrow$ need $v_y(y=0) = 0$

$$v_y = \frac{\Gamma x}{x^2 + (y-a)^2} - \frac{\Gamma x}{x^2 + (y+a)^2} = 0 \text{ at } y = 0.$$

Hence we can deduce motion near a wall. (add arrows)

In $d=3$, we have vortex lines, not point vortices.



What are the dynamics of vortex lines?

Analogy to Biot-Savart Law: for one vortex line

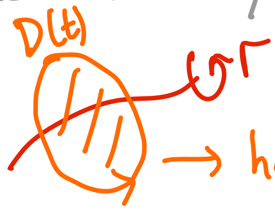
$$\vec{v}(\vec{x}) = \frac{\Gamma}{4\pi}\int ds\, \frac{\partial_s \vec{r} \times (\vec{x}-\vec{r}(s))}{|\vec{x}-\vec{r}(s)|^3}$$

Analogy to 2d case:

$$\partial_t \vec{r}(s,t) = \vec{v}(\vec{r}(s,t))$$

so vortex line follows the field it generates.

One nice way to derive this is to derive Kelvin's circulation Thm.



→ have points in D follow local velocity field

$$\frac{d}{dt}\int_{\partial D(t)} d\vec{r}\cdot\vec{v} = \text{“}\frac{d}{dt}\Gamma\text{”} \text{ using Stoke's Thm: } = \int_{D(t)} d\vec{A}\cdot\vec{\omega}$$

$$= \oint_{\partial D} \vec{v}\cdot\frac{d\vec{r}}{dt} + \oint_{\partial D}\Big[\partial_t + \Big(\underbrace{\frac{d\vec{r}}{dt}}_{=\vec{v}}\cdot\nabla\Big)\Big]\vec{v}\cdot d\vec{r}$$

neglect dissipation

$$= \oint_{\partial D} \underbrace{\vec{v}\cdot d\vec{v}}_{=\frac{1}{2}d(v^2)} + \oint_{\partial D}\Big[\cancel{\nu\nabla^2\vec{v}} - \frac{1}{\rho}\nabla P\Big]\cdot d\vec{r}$$

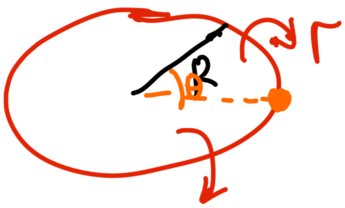
$= 0$ since we're integrating gradient around closed loop

↳ Kelvin's Circulation Theorem: $\displaystyle \frac{d}{dt}\oint_{\partial D(t)} \vec{v}\cdot d\vec{r} = 0$

Now if we watch a vortex line evolve in time we see:

vortex must have followed $\vec{v}$

Example 4: vortex ring.



Calculate velocity on ring:

$$v_z = \frac{\Gamma}{4\pi} \int_0^{2\pi} (R\,d\theta)\, \frac{\hat{\theta} \times (R\hat{x} - R\hat{r})}{R^3 |\hat{x} - \hat{r}|^3}$$

$$= \frac{\Gamma}{4\pi R} \int_0^{2\pi} d\theta\, \frac{-\hat{z}(1-\cos\theta)}{(2-2\cos\theta)^{3/2}}$$

This integral is divergent and we can estimate it as: $\log \frac{R}{a}$

a ← microscopic core size.

Hence vortex ring propagates at a constant velocity.