PHYS 7810
Hydrodynamics
Spring 2026

Lecture 12
Incompressible viscous flows

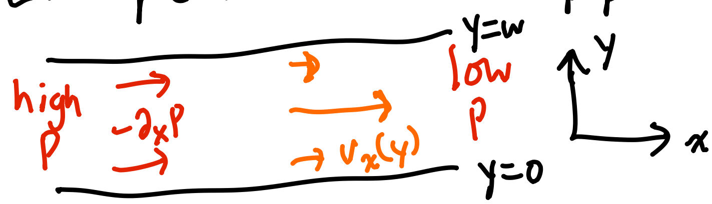
February 17

Assume incompressible flow ($\nabla \cdot \vec{v} = 0$) w/ ρ & T "constant":

$$\partial_t v_i + v_j \partial_j v_i + \frac{1}{\rho}\partial_i P = \frac{\eta}{\rho}\partial_j \partial_j v_i = \nu \partial_j \partial_j v_i$$

Today we'll look at some exact solutions highlighting the role of viscosity in flow patterns.

Example 1: Flow in pipe (Poiseuille flow) $(d=2)$



Assume no-slip boundary conditions: $\vec{v} = 0$ on walls.

Assume static ($\partial_t = 0$) and homogeneous ($\partial_x \vec{v} = \vec{0}$, $-\partial_x P = \text{const.}$)

Ansatz: $v_x(y)$

$$\hookrightarrow \cancel{v_x \partial_x v_x} + \underbrace{\frac{1}{\rho}\partial_x P}_{\text{const.}} = \nu \nabla^2 v_x = \nu \partial_y^2 v_x$$

Solve this ODE easily! $v_x(y) = \underbrace{c_1 + c_2 y}_{\text{integration const.}} + \frac{y^2}{2}\frac{\partial_x P}{\eta}$

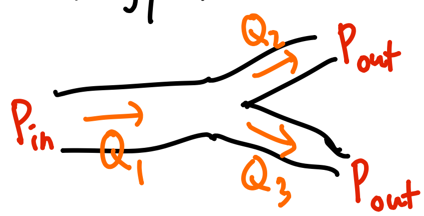
$$v_x(y=0)=0 \Rightarrow c_1=0$$
$$v_x(y=w)=0 \Rightarrow c_2 = -\frac{w}{2}\frac{\partial_x P}{\eta}$$
$$\rightarrow v_x(y) = -\frac{\partial_x P}{2\eta}\, y(w-y)$$

Total fluid flow: (we'll ignore constant factor of ρ for simplicity)

$$Q = \int_0^w dy\ v_x(y) = \frac{|\partial_x P|}{2\eta}\int_0^w dy\ y(w-y) = \frac{|\partial_x P|}{2\eta}\left(\frac{w^3}{2}-\frac{w^3}{3}\right) = \frac{|\partial_x P|}{12\eta} w^3$$

would scale as w^4 in 3d

Analogy to electrical resistors:



Mass conservation: $Q_1 = Q_2 + Q_3$

"Ohm's Law": $\Delta P = Q \cdot R$

"hydraulic resistance"

So...

electric	fluid
voltage	pressure
current	Q
R	R

for pipe of length L:

$$|\partial_x P| = \frac{\Delta P}{L}, \text{ so } R = \frac{12\eta L}{w^3}$$

Notice that geometric scaling of pipe resistance vs. geometry is very different vs. ordinary electrical resistors. This is a potential diagnostic to look for viscous flow of electrons!

On HW3 you will look at an application of such "resistor networks" to the circulatory system.

Exact solutions are hard to find. So working w/ dimensionless parameters is a good way to identify asymptotic/simplifying regimes...

"Lagrange multiplier for $\partial_i v_i = 0$"

$$\frac{v_{typ}^2}{\ell_{typ}} \leftarrow \underbrace{v_j \partial_j v_i} + \frac{1}{\rho}\partial_i P = \underbrace{\nu\, \partial_j \partial_j v_i} \rightarrow \frac{\nu\, v_{typ}}{\ell_{typ}^2} \qquad \left(\partial_i \sim \frac{1}{\ell_{typ}}\right)$$

compare

Here we're being heuristic and thinking in terms of "typical" length and velocity scales.

Reynolds number $\mathcal{R} = \frac{v_{typ}\, \ell_{typ}}{\nu}$ characterizes relative importance.

$\mathcal{R} \ll 1$: "creep flow" viscosity dominates

$\mathcal{R} \gg 1$: $\approx$inviscid flow

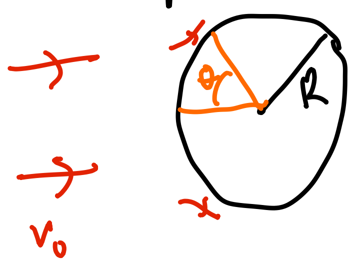
Which regime are we in for everyday life?

$$v_{typ} \sim 1\ m/s, \quad \ell_{typ} \sim 1\ m/s, \quad \nu \sim 10^{-5}\ m^2/s \rightarrow \mathcal{R} \sim 10^5$$

for "ordinary life"

But since nonlinear equations are harder to solve let's first go to the creep flow regime for the rest of the day...

Example 2: viscous drag on sphere in $d=3$. $(\mathcal{R} \ll 1)$



$$\cancel{v_j \partial_j v_i} + \frac{1}{\rho}\partial_i P = \nu \partial_j \partial_j v_i$$

curl:

$$\nu \nabla^2 (\nabla \times \vec{v}) = 0$$

$\nabla \cdot \vec{v} = 0$ means $\vec{v} = \nabla \times \vec{A}$

Try $\vec{A} = \psi(r,\theta)\,\hat{\phi}$

stream function!

This guess is motivated by fact that $\vec{v}$ shouldn't depend on ϕ.

EOM becomes: $\nu \nabla^2(\nabla^2 \vec{A}) = \vec{0} \rightarrow \nabla^2\nabla^2(\psi\hat{\phi}) = 0 \rightarrow \left(\nabla^2 - \frac{1}{r^2 \sin^2\theta}\right)^2 \psi = 0$

Need to be careful b/c Laplacian was acting on a vector!

Boundary conditions: ① $\vec{v} \rightarrow \vec{v}_0$ as $r \rightarrow \infty$

② $\vec{v} = \vec{0}$ at $r = R$

Since $\vec{v}_0 = -v_0\hat{z} = -v_0(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$, need

$[\nabla \times (\psi\hat{\phi})]_\theta \rightarrow v_0 \sin\theta$: $\quad v_0 \sin\theta = \frac{1}{r}\partial_r(r\psi)$? $\rightarrow \psi \rightarrow \frac{1}{2} v_0 r \sin\theta$

Using spherical symmetry we can now guess that we should be expanding ψ into spherical harmonics. In particular:

$\psi(r,\theta) = f(r)\sin\theta$:

$$\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)^2 (f(r)\sin\theta) = 0 \Rightarrow \left(\frac{1}{r^2}\partial_r r^2 \partial_r - \frac{2}{r^2}\right)^2 f = 0$$

Ansatz: $f(r) = r^a$? $\rightarrow 0 = \big((a-2)(a-1)-2\big)\big(a(a+1)-2\big)$

Solved by $a = 1, -2, 3, 0$: $f(r) = c_1 r + \frac{c_2}{r^2} + \cancel{c_3 r^3} + c_4$.

inconsistent w/ $r \to \infty$

$$\vec{v} = \nabla \times (\psi \hat{\phi}) = \frac{2f}{r}\cos\theta\,\hat{r} - \underbrace{\frac{1}{r}\partial_r(rf)\sin\theta\,\hat{\theta}}_{\text{large } r \text{ asymptotics: } c_1 = -\frac{v_0}{2}}$$

No-slip at $r=R$: $f(R) = 0 = -\frac{v_0 R}{2} + \frac{c_2}{R^2} + c_4$ $[v_r = 0]$

$\partial_r(rf)\big|_{r=R} = 0 = -v_0 R - c_2/R^2 + c_4$ $[v_\theta = 0]$

Solve: $c_4 = \frac{3}{4} v_0 R$, $c_2 = -\frac{v_0 R^3}{4}$.

Net drag force: $\underbrace{-F_i}_{} = \int_{\text{fluid}} d^3x\, \partial_t g_i = -\oint_{r=R} dA_j \tau_{ji}$

Newton's 3rd Law: F_i is the force on object

Use rotational symmetry: $F_z = \oint_{r=R} dA \cdot (-\tau_{rz})$ mixing coord systems...

b/c bdy of fluid points inward!

$$\tau_{ij} = P\delta_{ij} - \eta[\partial_i v_j + \partial_j v_i]$$

$$\rightsquigarrow \tau_{rz} = \cos\theta \cdot P - \eta\left[\cos^2\theta\, \partial_r\left(\frac{4f}{r}\right) + \sin^2\theta\left[\frac{2f}{r^2} + \partial_r\left(\frac{1}{r}\partial_r(fr)\right)\right]\right]$$

To evaluate this expression: $\partial_z = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_r$

$v_z = v_r\cos\theta - v_\theta\sin\theta$

Evaluate at $r=R$: $\tau_{rz} = P\cos\theta - \sin^2\theta\cdot\frac{3\eta}{2}\frac{v_0}{R}$

Some intermediate algebra: $f(r) = -\frac{v_0 r}{2} + \frac{3}{4}v_0 R - \frac{v_0 R^3}{4r^2}$

$$4\,\partial_r\left(\frac{f}{r}\right)\Big|_{r=R} = -\frac{3v_0R}{R^2} + \frac{3v_0R^3}{R^4} = 0,$$

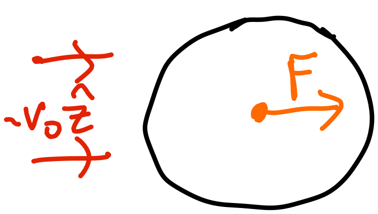
$$\partial_r\left(\frac{1}{r}\partial_r(fr)\right)\Big|_{r=R} = \partial_r\left(-v_0 + \frac{3}{4}\frac{v_0R}{r} + \frac{v_0R^3}{4r^3}\right)\Big|_{r=R} = -\frac{3}{4}\frac{v_0R}{R^2} - \frac{3v_0R^3}{4R^4} = -\frac{3v_0}{2R}$$

Calculate pressure P: $\eta\nabla^2\vec{v} = \nabla P$

After a tedious calculation one finds: $P(r,\theta) = \frac{3}{2}\eta v_0 R\frac{\cos\theta}{r^2}$

$$F_z = -\int_0^\pi d\theta\int_0^{2\pi} d\phi\cdot R^2\sin\theta\left[\underbrace{\frac{3\eta v_0}{2R}\cos^2\theta}_{P} + \underbrace{\frac{3}{2}\sin^2\theta\,\eta\frac{v_0}{R}}_{\eta\nabla v}\right]$$

$$= -2\pi R^2\cdot\frac{3\eta v_0}{2R}\int_0^\pi d\theta\,\sin\theta = -6\pi\eta v_0 R$$



force tries to align object motion w/ fluid

So far we have assumed that Reynolds number was small. Check if that assumption was accurate?

Method of dominant balance:

$$\rho\vec{v}\cdot\nabla\vec{v} + \nabla P = \eta\nabla^2\vec{v}$$

large terms should cancel

correction should be negligible...

$\nabla P \sim \frac{\eta v_0 R}{r^3}$ as $r\to\infty$ $\qquad \rho(\vec{v}_0\cdot\nabla)\vec{v} \sim v_0\partial_z\vec{v} \sim \frac{\rho v_0^2 R}{r^2}$ as $r\to\infty$

So our approximations are <u>inconsistent</u> at large r!

We can estimate where the creep flow approximation fails by comparing magnitudes of 2 terms:

$$\frac{\eta v_0 R}{r^3} \sim \frac{v_0^2 R \rho}{r^2} \quad \text{when} \quad \frac{\nu}{r} \sim v_0 \quad \text{or} \quad r \sim \frac{\nu R}{v_0 R} \sim \frac{R}{\mathcal{R}} \leftarrow \text{Reynolds}$$

Strategy of matched asymptotic expansions: