PHYS 7810
Hydrodynamics
Spring 2026

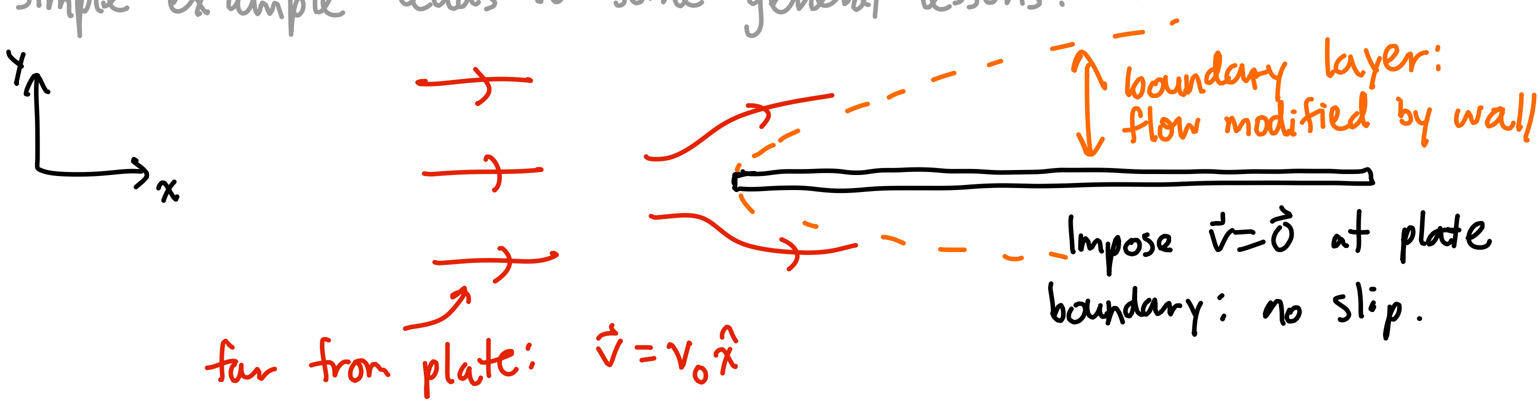
Lecture 13
Boundary layers

February 19

Incompressible fluid, static flow:

$$\nabla\cdot\vec{v}=0 \qquad (\vec{v}\cdot\nabla)\vec{v}+\frac{\nabla P}{\rho}=\nu\nabla^2\vec{v}$$

What is the flow around a thin plate? Today we'll see how this simple example leads to some general lessons. Assume $d=2$.



Our expectation is that we form a boundary layer!

If $\nu=0$: $\vec{v}=v_0\hat{x}$ solves all boundary conditions

↳ $\nu>0$ crucial for boundary layer.

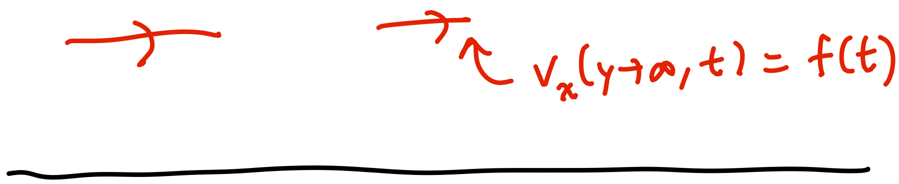
What's the Reynolds number?

$$\mathcal{R}=\frac{v_0 L}{\nu}$$ ↳ no length scale L!

This flow may be rather nonlinear...

Before turning to nonlinear Navier-Stokes equations, let's warm-up w/ a simpler problem:

Infinite plate:



Translation invariance: $\partial_x = 0$

Incompressible: $\partial_x v_x + \partial_y v_y = 0 = \partial_y v_y$

$\hookrightarrow v_y = 0$ by boundary condition

this drives flow!

$$\partial_t v_x + \cancel{v_x \partial_x v_x} + \cancel{v_y \partial_y v_x} + \boxed{\frac{\partial_x p}{\rho}} = \nu \partial_y^2 v_x$$

Fourier transform in t:

$$-i\omega v_x + c(\omega) = \nu \partial_y^2 v_x$$

$c(\omega)$: TBD

$$\hookrightarrow v_x(y) = \frac{ic}{\omega} + A e^{\sqrt{-i\omega/\nu}\, y} + B e^{-\sqrt{-i\omega/\nu}\, y}$$

$\frac{ic}{\omega}$: fix at $y=\infty$

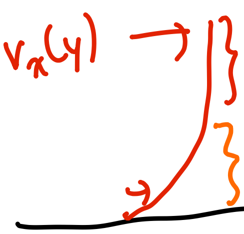
$A=0$ by regularity

fix B at $y=0$.

$$\sqrt{-i} = e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$$

Deduce: $v_x(y) = \frac{ic}{\omega}\left[1 - e^{-\sqrt{\omega/\nu}\, y \cdot \frac{1-i}{\sqrt{2}}}\right]$

where $\tilde{f}(\omega) = ic/\omega$ ← F.T. of boundary condition $f(t)$



} effectively irrotational / $\nu=0$ flow

} boundary layer of thickness $\delta_b = \sqrt{\nu/\omega}$

So the boundary layer indeed vanishes as $\nu \to 0$. Now let's return to our original problem. We have no explicit t-dependence. But we do have x-dependence... so...

Effective boundary layer $\delta_b(x) \sim \sqrt{\frac{\nu}{\omega_{eff}(x)}} \sim \sqrt{\frac{\nu x}{v_0}}$?

This last estimate came from dimensional analysis.

Goal: approximate solution to equations for boundary layer.

Guess 1: $\nabla P \approx 0$ (we'll check it later)

$\hookrightarrow (\vec{v}\cdot\nabla)\vec{v} = \nu \nabla^2 \vec{v}$

stream function:

And, $\nabla\cdot\vec{v} = 0$ implies in 2d: $v_i = \varepsilon_{ij}\partial_j \psi$ $\begin{bmatrix} v_x = \partial_y \psi \\ v_y = -\partial_x \psi \end{bmatrix}$

Guess 2: $\frac{1}{x} \sim \partial_x \ll \partial_y \sim \frac{1}{\delta_b(x)}$ (at large x)

$v_y \ll v_x$

Important equation: $v_x \partial_x v_x + v_y \partial_y v_y \approx \nu \partial_y^2 v_x$

$$\boxed{\partial_y \psi\, \partial_x \partial_y \psi - \partial_x \psi\, \partial_y^2 \psi \approx \nu \partial_y^3 \psi}$$

boundary layer equation

In general solving nonlinear PDEs is hard so let's work w/ dimensionless variables?

$[v_0] = \frac{[L]}{[T]}$ $\qquad [\psi] = \frac{[L]^2}{[T]} = [\nu]$ $\qquad [x] = \frac{1}{[\partial_x]} = [L]$

$\hookrightarrow \Psi = \psi/\nu \qquad (X, Y) = \frac{v_0}{\nu}(x, y)$

$$\partial_Y \Psi\, \partial_X \partial_Y \Psi - \partial_X \Psi\, \partial_Y^2 \Psi = \partial_Y^3 \Psi$$

Guess 3: Similarity ansatz: $\Psi(X,Y) = X^\alpha f\left(\frac{Y}{X^\beta}\right)$

Exponents α, β and function f to be found.

Call this ξ

The ansatz is inconsistent for most values of α:

$$\partial_Y^3 \Psi = X^{\alpha-3\beta} f'''(\xi)$$

$$\partial_X \Psi \, \partial_Y^2 \Psi = \left[\alpha X^{\alpha-1} f(\xi) - \beta \frac{\xi}{X} X^{\alpha} f'(\xi)\right]\left[X^{\alpha-2\beta} f''(\xi)\right]$$

$$\sim X^{2\alpha-2\beta-1}\left[\xi\text{-dependent}\ldots\right]$$

consistent if: $\alpha - 3\beta = 2\alpha - 2\beta - 1$ or $\alpha = 1-\beta$.

This is true for generic boundary layers. For our particular problem we also have a boundary condition

$$v_x(y \to \infty) = v_0$$

$\hookrightarrow$ $v_0 \partial_Y \Psi \big|_{\xi=\infty} = v_0 \rightarrow \partial_Y \Psi = X^{\alpha-\beta} f'(\xi)$

$\alpha = \beta = 1/2$ $\qquad$ $f'(\infty) = 1$

Now we take our boundary layer equation:

$$\partial_Y^3 \Psi = \partial_Y \Psi \, \partial_X \partial_Y \Psi - \partial_X \Psi \, \partial_Y^2 \Psi$$

$\hookrightarrow$ $\frac{1}{X} f'''(\xi) = \frac{1}{X}\left[f' \cdot \left(-\frac{\xi}{2} f''\right) - \frac{1}{2}(f - \xi f') f''\right]$

$\hookrightarrow$ $\boxed{f''' = -\frac{1}{2} f f''}$ $\quad$ Blasius' equation (1907)

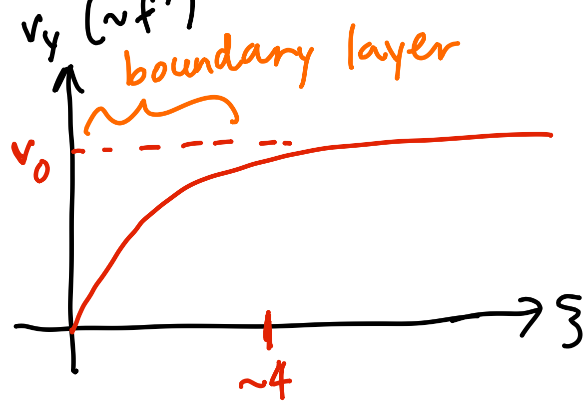
Solve for boundary conditions: $\quad f'(\infty) = 1$

$v_x = v_0 \partial_Y \Psi = 0$ at $y = 0$: $\quad f'(0) = 0$

$v_y = -v_0 \partial_X \Psi = 0$ at $y = 0$: $\quad \frac{1}{2\sqrt{X}}\left[f - \xi f'\right]\Big|_{\xi=0} = 0$

$\hookrightarrow$ $f(0) = 0$.

Numerical solution:



The physics ends up very similar to what we found in our toy example.

Now we need to go back and check that our approximations are accurate!

Check: $\underbrace{|\partial_y \Psi|}_{v_x} \underbrace{\gg}_{\gg} \underbrace{|\partial_x \Psi|}_{v_y} \rightsquigarrow f'(\xi) \gg \frac{1}{2\sqrt{X}}[f - \xi f']$

OK if $X \gg 1$

$$X = x\, v_0/\nu \gg 1 \quad \text{or} \quad x \gg \nu/v_0$$

Since $\delta_b \sim \sqrt{\frac{\nu x}{v_0}}$, we need $\delta_b \ll x$

The boundary layer approx fails near the tip $x=0$.

Check: pressure gradients small?

↳ estimate from the other equation:

$$\underbrace{v_x \partial_x v_y + v_y \partial_y v_y}_{\sim \frac{v_0}{x} v_y} + \frac{\partial_y P}{\rho} \approx \underbrace{\nu \partial_y^2 v_y}_{\nu/\delta_b^2\, v_y} \rightarrow \text{equal scalings!}$$

method of dominant balance

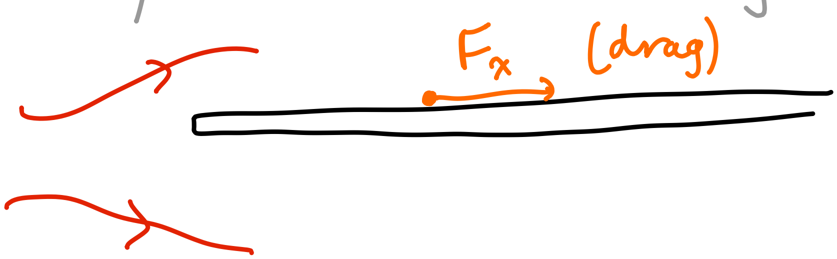
Therefore $\frac{\partial_y P}{\rho} \lesssim \frac{\nu}{\delta_b^2} v_y$ and $\frac{\Delta P}{\rho} \lesssim \frac{\nu}{\delta_b} v_y \sim \nu \frac{v_0}{x} \sim \left(\frac{\delta_b v_0}{x}\right)^2$

The term we neglected when calculating boundary layer equation:

$$\frac{\partial_x P}{\rho} \sim \frac{1}{x}\left(\frac{\delta_b v_0}{x}\right)^2 \sim \left(\frac{\delta_b}{x}\right)^2 v_0 \cdot \frac{1}{x} v_0 \sim \underbrace{\left(\frac{\delta_b}{x}\right)^2}_{\text{small correction once } \delta_b \ll x} v_x \partial_x v_x$$

Hence our approximations were accurate!

Lastly let's estimate the drag force on our plate (per unit width):



$$\frac{F_x}{\underset{\text{width}}{w}} = \underset{\text{top \& bottom}}{2}\int_0^L dx\,(-\tau_{yx}) = 2\eta\int_0^L dx\,\partial_y v_x = 2\eta\int_0^L dx\,\partial_y^2\psi$$

$$= 2\eta\int_0^L d\left(\frac{\nu}{v_0}X\right)\cdot\left(\frac{v_0}{\nu}\partial_Y\right)^2 \nu\Psi$$

$$= 2\nu\rho v_0\int_0^{v_0L/\nu} dX\,\frac{f''(0)}{\sqrt{X}} = 4\rho\sqrt{\nu v_0^3 L}\cdot \overset{\text{some const.}}{f''(0)}$$

For $L \gg \nu/v_0$ we have:

$$F_x = 4\rho v_0^2 L w\cdot\sqrt{\frac{\nu}{v_0 L}}\, f''(0) = 2\rho v_0^2 \underset{\text{surface area}}{A}\cdot\frac{1}{\sqrt{R}}\, f''(0)$$

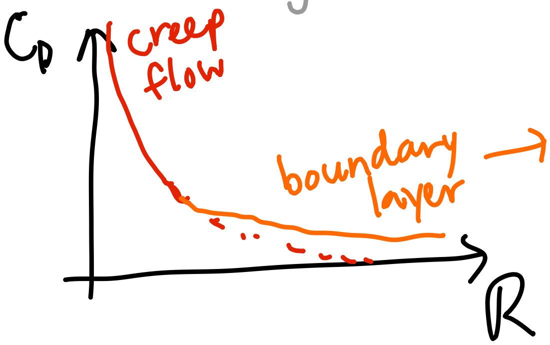
Rescale as: $$F_x = \frac{1}{2}\rho v_0^2 A\cdot \underset{\text{drag coefficient}}{C_D}$$

For plate: $C_D \approx 1.3\, R^{-1/2}$

Compare to creep flow (Lec 12, for sphere...):

$$F = 6\pi\rho\nu R v_0 = \frac{\rho v_0^2}{2} \cdot \underbrace{(4\pi R^2)}_{A} \cdot \frac{3}{\mathcal{R}} \quad \text{where } \mathcal{R} = \frac{v_0 R}{\nu}.$$

Hence in general we predict:



→ increases drag!