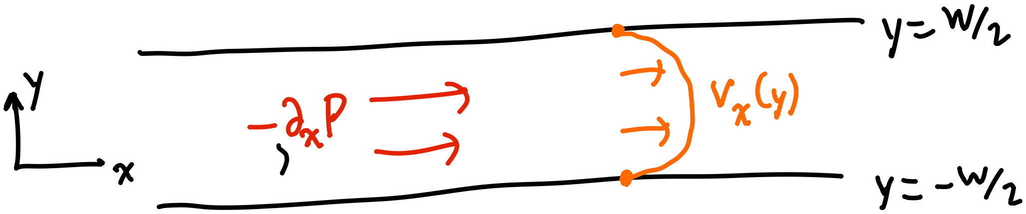
PHYS 7810
Hydrodynamics
Spring 2026

Lecture 14
Instabilities of viscous flows

February 24

Recall Poiseuille flow (lec 12): in $d = 2$



$$v_x(y) = v_{max}\left[1 - \left(\frac{2y}{W}\right)^2\right]$$

This is an exact solution of nonlinear incompressible Navier–Stokes equations for any v_{max}. But to observe this flow in nature it must be a stable solution of N–S...

Consider perturbing our ideal solution:

$$P = P_0 + \varepsilon P_1 + \cdots \qquad \vec{v} = \vec{v}_0 + \varepsilon \vec{v}_1 + \cdots$$

ideal Poiseuille flow

generic small perturbation ($\varepsilon \to 0$)

What happens at $t \to \infty$? In particular, will small perturbations grow large w/ time, thus eventually becoming comparable to our initial solution?

Goal: determine stability of flow (in "linear response")

Navier-Stokes: $\partial_t \vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \frac{\nabla P}{\rho} - \nu\nabla^2\vec{v} = 0$

$$\left[\partial_t \vec{v}_0 + (\vec{v}_0\cdot\nabla)\vec{v}_0 + \frac{\nabla P_0}{\rho} - \nu\nabla^2\vec{v}_0\right] \leftarrow \text{automatically } = 0$$

$$+\,\varepsilon\underbrace{\left[\partial_t \vec{v}_1 + (\vec{v}_0\cdot\nabla)\vec{v}_1 + (\vec{v}_1\cdot\nabla)\vec{v}_0 + \frac{\nabla P_1}{\rho} - \nu\nabla^2\vec{v}_1\right]} + \cdots = 0$$

need to solve this equation.

If flow stays incompressible, $\nabla\cdot\vec{v}_1 = 0$.

Let $\vec{v}_0 = U(y)\hat{x}$.

↳ Independent of x & t, so Fourier transform in x&t:

$\vec{v}_1(y)e^{ikx-i\omega t}$ and $P_1(y)e^{ikx-i\omega t}$

↳ introduce stream function: $v_{1i} = \varepsilon_{ij}\partial_j\psi_1$

Then:

$$ik\frac{P_1}{\rho} = \nu\left[(\partial_y^2 - k^2)\partial_y\psi_1 + i\omega\partial_y\psi_1 + ik\psi_1 U' - ikU\partial_y\psi_1\right]$$

$$\partial_y\left(\frac{P_1}{\rho}\right) = \nu\left[(\partial_y^2 - k^2)(-ik\psi_1) + i\omega(-ik\psi_1) - ikU(-ik\psi_1)\right]$$

Combine:

$$-i\omega\underbrace{(\partial_y^2 - k^2)}_{B}\psi_1 = \underbrace{\left[-ikU(\partial_y^2 - k^2) + ikU'' + \nu(\partial_y^2 - k^2)^2\right]}_{A}\psi_1$$

This is a generalized eigenvalue problem:

$$\hat{A}\psi_1 = -i\omega\hat{B}\psi_1$$

It needs to be solved numerically. The key point is that a typical eigenvalue is complex-valued:

$$\omega = \omega' + i\omega'' \qquad (\omega', \omega'' \text{ are real})$$

Then $\psi_1 \to \psi_1(y)\, e^{ikx}\, e^{-i\omega' t}\, \boxed{e^{\omega'' t}}$

Instability if $\omega'' > 0$, or $\mathrm{Im}(\omega) > 0$

NB: we take k to be real b/c we don't (in this problem) want perturbations to be large near $x = \pm\infty$. But we only fix perturbations at $t=0$... they can diverge as $t \to \infty$.

Also note that instability here is caused by boundary conditions. The fluid is still trying to reach thermal equilibrium, consistent with our derivation of hydro EFT.

Work in dimensionless units:

$$U(y) = v_{max}\left(1 - \underbrace{\left(\frac{2y}{w}\right)}^{2}\right)$$

$$\tilde{y} = \frac{2y}{w}, \quad \tilde{k} = \frac{w}{2}k, \quad \tilde{\omega} = \frac{w}{2v_{max}}\omega$$

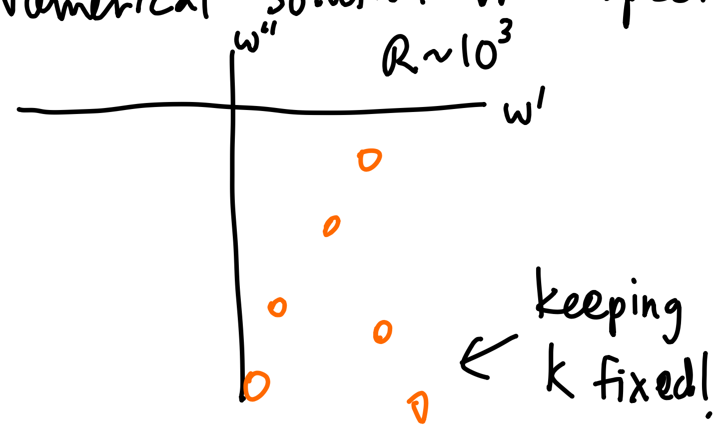
and the Reynolds number $R = \dfrac{v_{max} w}{2\nu}$.

Then our linear response equation becomes the Orr-Sommerfeld equation: (no tildes)

$$-i\omega(\partial_y^2 - k^2)\psi_1 = \left[-ik(1-y^2)(\partial_y^2 - k^2) - 2ik + \frac{1}{R}(\partial_y^2 - k^2)^2\right]\psi_1$$

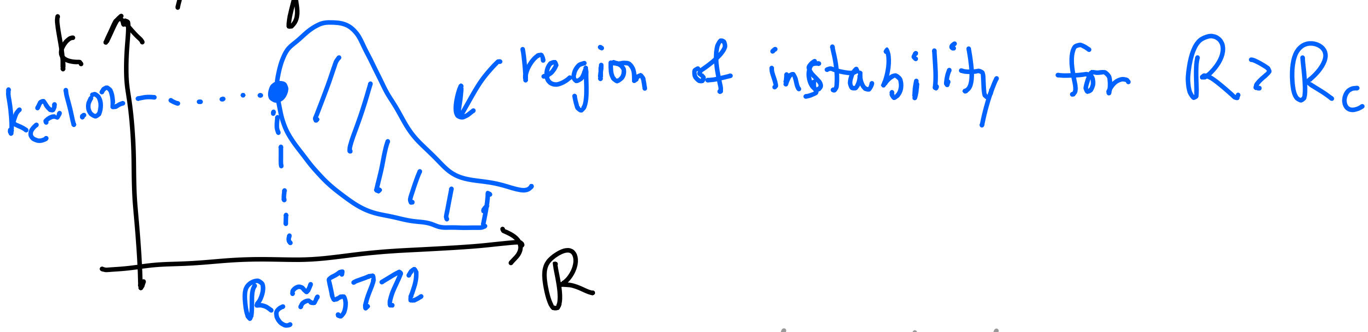
for $-1 \le y \le 1$.

Numerical solution via spectral methods [Orszag, 1972]



Spectrum is discrete because we're solving the eigenvalue problem in a discrete window.

If any eigenvalue has $\mathrm{Im}(\omega) > 0$, flow is unstable:



A realistic/typical initial condition will have some wave numbers in the unstable region, so we'd never see Poiseuille flow once $R > R_c$. So our expectation would then be that:

endpoint of dynamics
- $R > R_c$ → turbulence! (lec 15) multiscale/chaotic flow...
- $R < R_c$ → laminar flow (smooth, not turbulent) ?

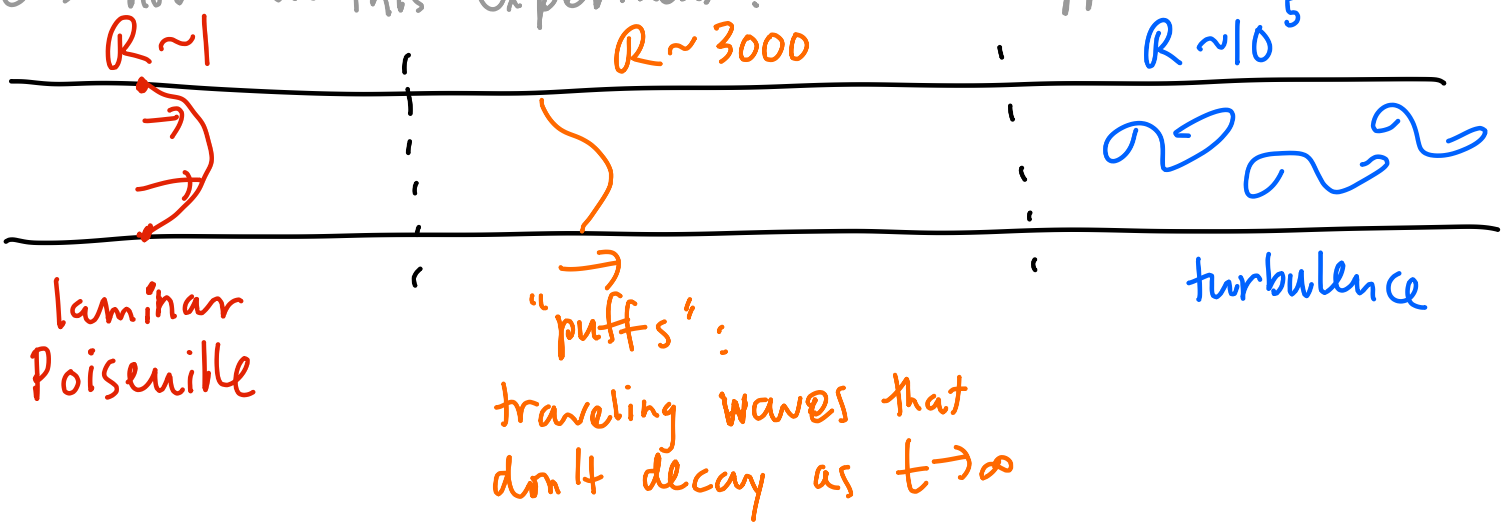
The fact that $R_c \sim 10^3 - 10^4$ is typical.

Everyday fluid flow is turbulent...!

$\nu \sim 10^{-5}$ (air)
$v_0 \sim 1\ \mathrm{m/s}$
$L \sim 1\ \mathrm{m/s}$

$$R \sim \frac{v_0 L}{\nu} \sim 10^5$$

Let's now "do this experiment". What happens?



How is it possible something interesting happened for $R < R_c$?
To answer this question, we need a little detour....

Consider $R = R_c + \delta$ ($\delta \to 0^+$, just above linear instability)

Guess: at fixed k, one unstable mode, many stable modes ...

Schematic: $\psi(t) \sim \left[A(t)\,\psi_1 + \sum_{n>1} B_n(t)\,\psi_n \right] e^{ikx}$

$$\dot{A}(t) = -i\omega_1 A + \text{nonlinear terms}$$

Guess: only A matters? $\rightsquigarrow$ $\dot{A} = -i\omega_1 A \underbrace{- \frac{\alpha_0}{2}|A|^2 A - \frac{\beta_0}{2}|A|^4 A + \cdots}$

Symmetry under $A \to Ae^{i\phi}$ (ϕ const.) fixes this

Better: $\frac{d}{dt}|A|^2 = 2\omega''|A|^2 - \alpha\,|A|^4 - \beta\,|A|^6 + \cdots$

Exact computation of α, β not realistic. The key thing is the qualitative argument...

Just above instability, expect $\omega'' = c \cdot \delta$ ($c > 0$)

Guess: $\alpha > 0$.

$d/dt\,|A|^2 = 0$ when $2c\delta|A|^2 \approx \alpha|A|^4$ or $|A| = \sqrt{\frac{2c\delta}{\alpha}}, 0$

This scaling is typical of a "mean-field transition". The key point is that the endpoint of the instability is not far away, once nonlinearity is accounted for.

What about $\alpha < 0$?

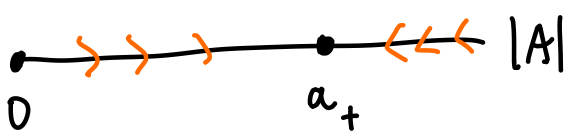
$$0 = 2c\delta|A|^2 + |\alpha|\,|A|^4 - \beta|A|^6$$

$|A| = 0$ or $\sqrt{a_\pm}$ where $a_\pm = \left[\dfrac{|\alpha| \pm \sqrt{|\alpha|^2 + 8\beta c\delta}}{2\beta}\right]^{1/2}$

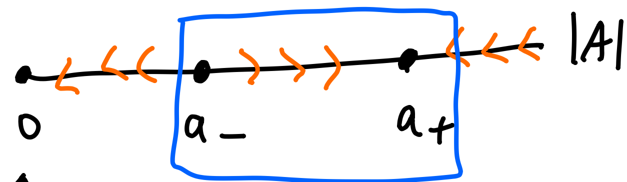
only real solutions acceptable!

The dynamics of $|A|$ is now more interesting...

$\delta > 0$:



$\delta < 0$:



This point is metastable!

puff dynamics for "large" perturbations?