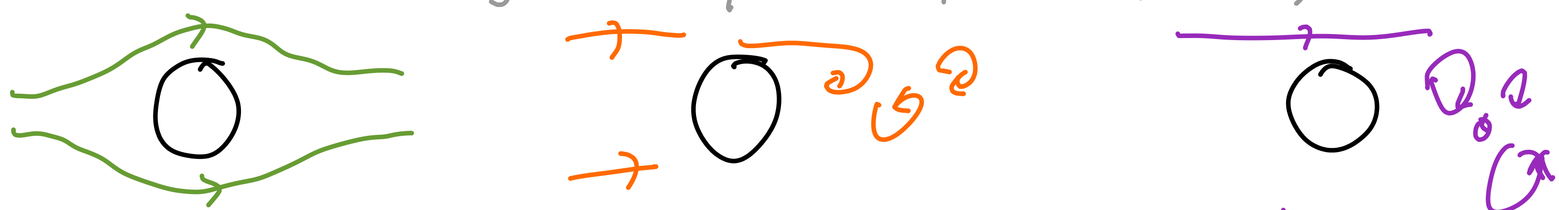
PHYS 7810
Hydrodynamics
Spring 2026

Lecture 15
Turbulence

March 3

Reynolds number $R = \frac{v_{typ}\, \ell_{typ}}{\nu}$ characterizes flow patterns...

We can illustrate this by thinking about the flow around an obstacle (assuming far away velocity is uniform...)



laminar flow (creep flow, $R \ll 1$)

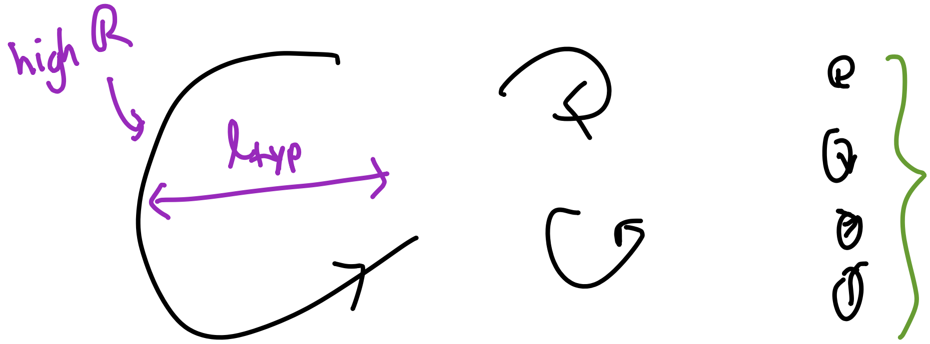
$R \sim 100$ nonlinear but predictable

$R \sim 10^4$ turbulence

Let's start by giving some high-level characterizations of turbulence.

Turbulent flow is:

- chaotic and multi-scale



smaller scale vortices w/ $R_{eff} \sim 1$

↳ dissipates energy

- characterized by direct cascade $(d=3)$:
  energy flow from big $\to$ small vortices

The picture to have in mind for turbulent flow is that a large vortex becomes unstable and splits into smaller vortices, which become smaller still... until at short scales viscosity damps out flow.

We don't have a rigorous theory for turbulence, so our understanding of subject is limited to simple cartoons/scaling arguments, or large scale numerics. Today, focus on cartoons...

Navier-Stokes: $\partial_t v_i + v_j \partial_j v_i + \frac{\partial_i P}{\rho} = \nu \partial_j \partial_j v_i$ and $\partial_i v_i = 0$

Write: $v_i = \bar{v}_i + \delta v_i$

large background flow ($\bar{v}_i$)

fluctuations/turbulence (δv_i)
↳ assume statistical description? $\langle \delta v_i \rangle = 0$

$$\partial_t \bar{v}_i + \bar{v}_j \partial_j \bar{v}_i + \frac{\partial_i P}{\rho} + \partial_j \underbrace{\langle \delta v_j \delta v_i \rangle}_{\text{square of fluctuations not small!}} = \underbrace{\cancel{\nu \partial_j \partial_j \bar{v}_i}}_{\text{negligible at large } R\,?}$$

Strategy #1: find EOM for $\langle \delta v_j \delta v_i \rangle$?

Take: $\left\langle \delta v_j \left[\partial_t v_i + v_k \partial_k v_i + \cdots \right] \right\rangle + (i \leftrightarrow j)$

$$= \partial_t \langle \delta v_i \delta v_j \rangle + \langle \delta v_j \delta v_k \rangle \partial_k \bar{v}_i + \langle \delta v_i \delta v_k \rangle \partial_k \bar{v}_i$$
$$+ \bar{v}_k \partial_k \langle \delta v_i \delta v_j \rangle + \frac{1}{\rho} \langle \delta v_j \partial_i \delta P \rangle + \frac{1}{\rho} \langle \delta v_i \partial_j \delta P \rangle = \cdots$$

Solve for: $\bar{v}_i$, P, $\langle \delta v_i \delta v_j \rangle$, $\langle \delta v_i \delta P \rangle$, higher moments...

This procedure is reminiscent of the way we will formally set up kinetic theory in Lecture 20.

While it's elegant in principle, in practice this approach is extremely annoying. So often one takes...

Strategy #2: $\underbrace{\text{"close EOMs"}}$ quickly

↳ phenomenology: $\boxed{\text{guess}}$ $\langle \delta v_i \delta v_j \rangle$ in terms of $\bar{v}$

"turbulent viscosity": $\langle \delta v_i \delta v_j \rangle \approx \nu_t (\partial_i \bar{v}_j + \partial_j \bar{v}_i)$

where $\nu_t \sim v_{typ} \ell_{typ}$?

↳ when $R \gg 1$, $\nu_t \gg \nu$ as you'd expect

Physical interpretation of ν_t is that it's accounting for multiscale vortex dynamics in turbulent flow...

Example: wake around disk ($d=2$):

smear out and deduce mean flow based on turbulent viscosity?

By dimensional analysis: drag force $F_x \sim C_D \times \frac{1}{2} \rho v_0^2 R$

drag coefficient: assume $O(1)$ for turbulent flow!

Far behind the object we might expect to use boundary layer theory:

$\vec{v} \approx v_0 \hat{x} + \delta \vec{v}$ and: $v_0 \partial_x \delta \vec{v} \approx \nu_t \nabla^2 \delta \vec{v} \approx \nu_t \partial_y^2 \delta \vec{v}$

We recognize the solution to this equation from lecture 2!

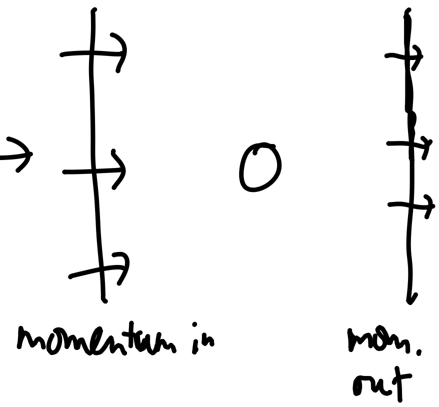
$$\delta v_x \approx \tilde{C} \cdot \frac{e^{-y^2/4\nu_t(x/v_0)}}{\sqrt{4\pi \nu_t x/v_0}}$$

integration constant TBD

We should take $\nu_t \sim \delta v_x(y=0)\cdot \Delta y \sim \frac{\tilde{C}}{\cancel{\Delta y}}\cdot\cancel{\Delta y}$ b/c of 1d Gaussian profile!

width of wake

$$F_x = \int_{-\infty}^{\infty} dy \left[\tau_{xx}(x=-\infty) - \tau_{xx}(x=+\infty)\right]$$



$$F_x \sim \int_{-\infty}^{\infty} dy \left[\rho v_0^2 - \rho(v_0+\delta v_x)^2\right] \approx -2\rho v_0 \int_{-\infty}^{\infty} dy\, \delta v_x$$

$$= -2\rho v_0 \tilde{C}.$$

But $F_x = \frac{C_D}{2}\rho v_0^2 R$ so $\tilde{C} = -\frac{C_D}{4} v_0 R$

So let's analyze the resulting wake using $C_D \sim 1$:

wake width: $\Delta y(x) \sim \sqrt{\nu_t\, x/v_0} \sim \sqrt{R x C_D}$ (purely geometric)

$\hookrightarrow \nu_t \sim v_0 R C_D$

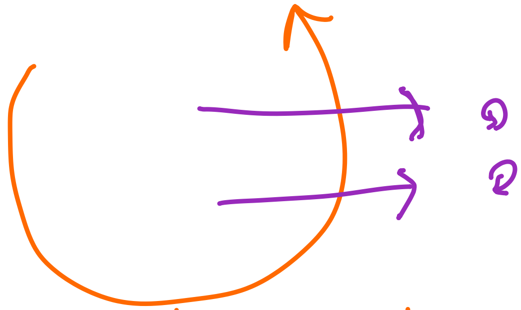
compare w/ laminar wake (on plate: Lec 13)

$$\delta_b \sim \sqrt{\frac{\nu x}{v_0}} \sim \frac{1}{\sqrt{\mathcal{R}}}\Delta y$$

The turbulent wake is much larger. It is "entraining" fluid from the outside into the wake. The fact that otherwise ambient fluid gets pulled into the turbulent wake greatly enhances the effect of drag in turbulent flow!

Kolmogorov's scaling theory of turbulence (1941):

Study isotropic, <u>driven</u> flow

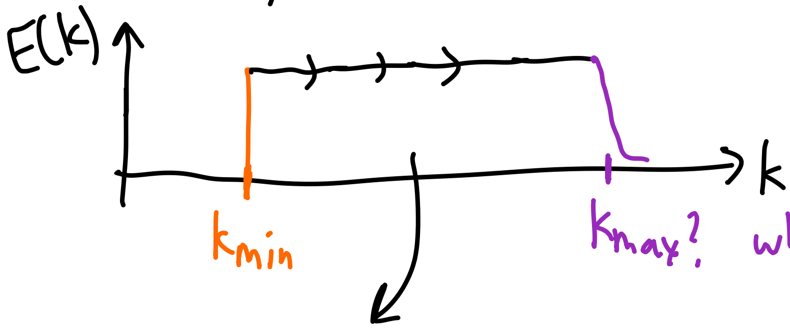


direct cascade of energy/vorticity to smaller scales

stir at large scale

$L_{stir} \sim \frac{1}{k_{min}}$

Statistical steady-state flow of energy in Fourier space!



when viscosity matters?

Postulate: uniform flow in $E(k) = \frac{\rho}{2}\int d^3q\, \delta(|q|-k)\langle v_i(-q) v_i(q)\rangle$

$$\sim \frac{\rho}{2} k^2 \underbrace{v_{typ}(k)}_{\text{typical velocity in eddy of size } 1/k}{}^2$$

Dimensional analysis: $[E(k)] \sim [\rho][k]^2[v(k)]^2$

$$\sim \frac{[M]}{[L]^3}\frac{1}{[L]^2}\underbrace{\left[\int d^3r\, e^{-ikr} v(r)\right]}_{\frac{[L]}{[T]}\cdot[L]^3}$$

So $[E(k)] \sim \frac{[M][L]^3}{[T]^2}$

Postulate: $E(k) \sim \underset{\text{total mass}}{M_{tot}}\, \varepsilon^\alpha k^\beta$ where $\varepsilon \sim \frac{1}{M}\left(\frac{dE}{dt}\right)_{stirring}$

$$[\varepsilon] \sim \frac{1}{[M]} \cdot \frac{[M][L]^2}{[T]^2} \cdot \frac{1}{[T]} \sim \frac{[L]^2}{[T]^3}$$

Now we combine scalings of $E(k)$ w/ ε to deduce α, β:

$$-2 = -3\alpha, \quad \text{so} \quad \alpha = 2/3$$

$$3 = 2\alpha - \beta = \frac{4}{3} - \beta \implies \beta = -5/3$$ Kolmogorov scaling

This can be readily observed in numerical simulations — it is a classic benchmark for observing turbulent flow.

Over what range of scales will turbulence exist?

Effective Reynolds number at each scale:

$$R(k) \sim \frac{v_{typ}(k)}{\nu k} \quad v_{typ}(k) \sim \left(\frac{\varepsilon}{k}\right)^{1/3}$$ by dimensional analysis

At stirring scale $R_{stir} \sim \dfrac{\varepsilon^{1/3}}{\nu k_{min}^{4/3}} \gg 1$

Viscosity destroys vortices at $k = k_{max}$ where $R(k_{max}) = 1$.

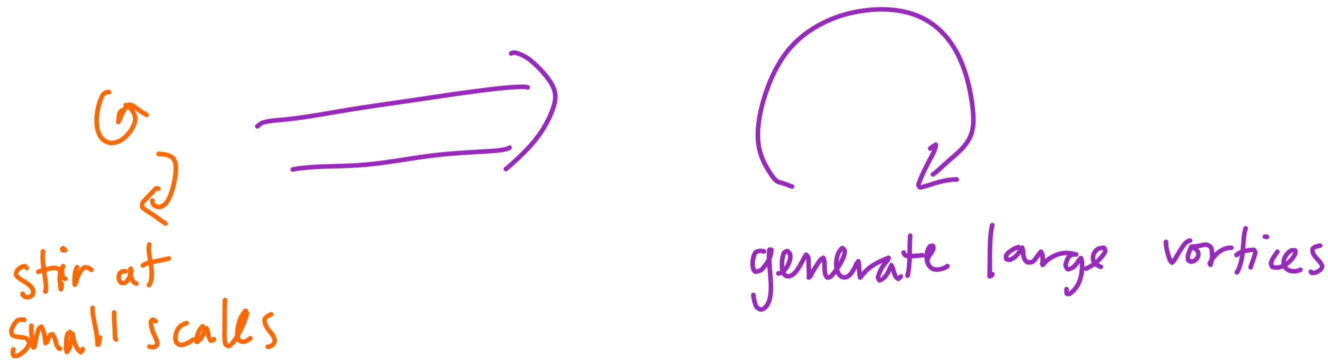
$$k_{max} \sim \frac{\varepsilon^{1/4}}{\nu^{3/4}}$$

And $\boxed{k_{max} \sim R_{stir}^{3/4} \cdot k_{min}}$

range of length scales over which turbulence observed!

Our analysis of turbulence has so far been entirely in $d=3$. In $d=2$, the Kolmogorov scaling is unchanged but one aspect of turbulence does qualitatively change...

$d=2$: <u>inverse cascade</u>:



A simple realization of this is in atmospheric flows where large hurricanes can form and last for a long time.
(Flow in atmosphere is quasi-2d $\rightarrow$ see lecture 17)
If there was a direct cascade, hurricanes would have been unstable!