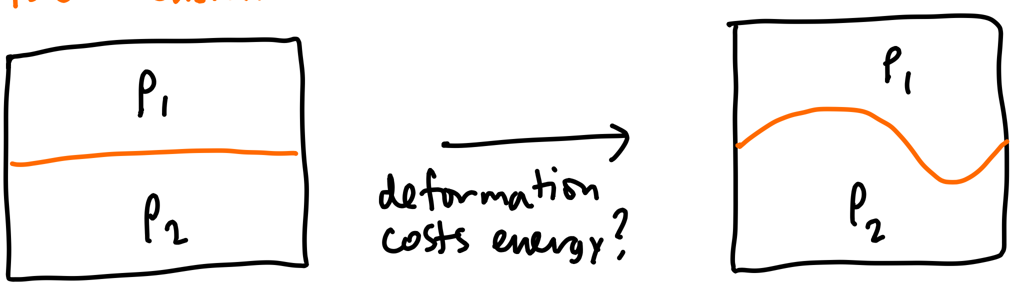
PHYS 7810
Hydrodynamics
Spring 2026

Lecture 16
Surface tension and gravity waves

March 5

Surface tension at interface btwn two fluids:

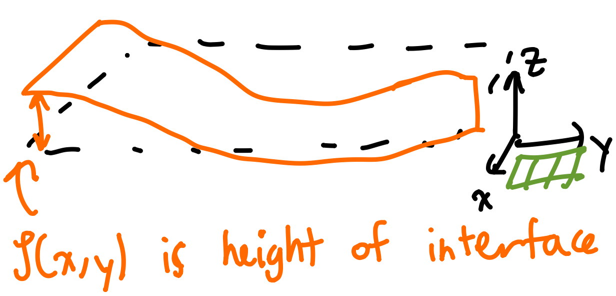


Postulate: change in potential energy $dU = \alpha \, dA$

$\alpha > 0$: coefficient of surface tension

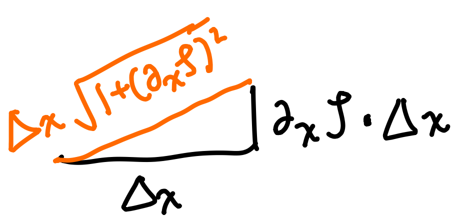
Our goal is to calculate how surface tension causes a modified boundary condition at the interface of 2 fluids.

At rest: interface at $z=0$.



Suppose $\nabla_\perp \zeta = (\partial_x \zeta, 0)$

with $|\partial_x \zeta| \ll 1$.



In generic coordinates; $dA = dx\,dy\sqrt{1+|\nabla\zeta|^2}$

$$\approx dx\,dy\left[1+\tfrac{1}{2}|\nabla\zeta|^2+\cdots\right]$$

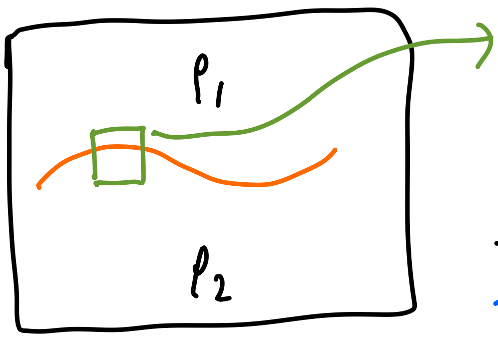
So the energy stored in surface tension is given by

$$U \simeq \alpha A = \alpha\int dx\,dy\left[\underbrace{\cancel{1}}_{\text{ignore}} + \tfrac{1}{2}|\nabla\zeta|^2\right] = \frac{\alpha}{2}\int dx\,dy\left[(\partial_x\zeta)^2+(\partial_y\zeta)^2\right]$$

Force per unit area acting on interface:

$$f_z = -\frac{\delta U}{\delta\zeta} = -\frac{\partial U}{\partial\zeta} + \nabla_\perp\cdot\frac{\partial U}{\partial\nabla_\perp\zeta} = \alpha\nabla_\perp^2\zeta = \alpha(\partial_x^2+\partial_y^2)\zeta$$

How does this force modify boundary conditions?



force balance in box?

Assume static configuration ($\vec{v}=\vec{0}$):

$$\cancel{\frac{d}{dt}\int_{\text{box}} d^3x\, g_z} + \oint_{\partial\text{box}} dA_j\,\tau_{jz} = f_z$$

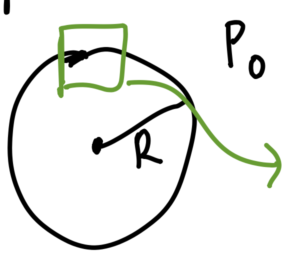
$$\downarrow$$

$$\simeq dA\left[P(z>\zeta)-P(z<\zeta)\right] = \alpha\nabla_\perp^2\zeta\cdot dA$$

So there's a pressure drop across the interface!

$$\Delta P = \alpha\cdot\underbrace{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}_{\text{radii of curvature near point}}\quad (=\nabla_\perp^2\zeta)$$

Example 1: bubble in air?



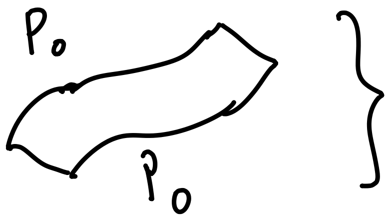
$$\zeta \approx -\frac{x^2+y^2}{2R}$$

Pressure inside bubble:

$$P_0 - P_{in} = -\frac{2\alpha}{R}$$

$$P_{in} = P_0 + \frac{2\alpha}{R}$$ so pressure inside bubble is higher.

Example 2: Soap film in air

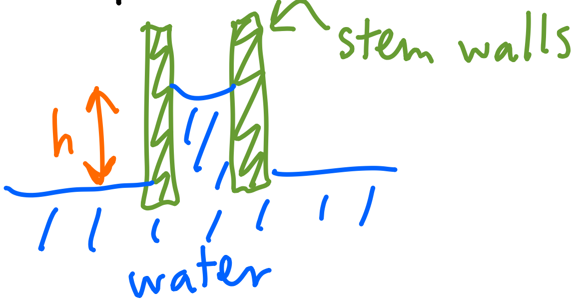


same ambient pressure P_0 on both sides:

$$\frac{1}{R_1} = -\frac{1}{R_2} \quad \text{or} \quad \boxed{\nabla_\perp^2 \zeta = 0}$$

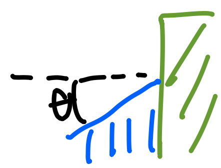
soap film solves Laplace's equation $\longrightarrow$ forms minimal area surface!

Example 3: capillary action in plants.



How high (h) does water rise?

Near cell wall we have:



This is a consequence of hydrophobic interactions btwn water and cell walls.

In presence of Earth's gravitational field we have:

$$\frac{1}{\rho}\nabla P = -g\hat{z} \longrightarrow P(z) = P_0 - \underbrace{\rho g z}_{\text{"hydrostatic pressure"}}$$

Pressure should be P_0 at all points above water:

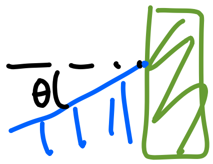
$$P = P_0 - \rho g(h + \cancel{\zeta})$$ ignore

From the surface tension condition we see that:

$$\alpha \nabla^2 \zeta \approx P_0 - P = \rho g h \xrightarrow{\text{assume cylindrical stem}} \alpha \frac{1}{r}\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right) = \rho g h$$

Solve: $\zeta(r) = \frac{\rho g h}{4\alpha} r^2$

Now let's relate ζ to the angle θ:



$$\sin\theta \approx \theta \approx \frac{d\zeta}{dr} = \frac{\rho g h}{2\alpha} R$$

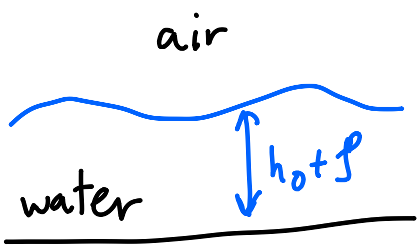
radius of stem.

microscopic property of interface

Jurin's Law: $h \approx \frac{2\alpha}{\rho g R} \cdot \theta$

This is a physical process by which (small) plants can carry water from the soil up to the leaves.

As a second illustration of a flow where surface tension matters we now turn to gravity waves.



Assume: – incompressible ($\nabla\cdot\vec{v}=0$)
– irrotational ($\nabla\times\vec{v}=\vec{0}$)

Then $\vec{v} = \nabla\Phi$ w/ $\nabla^2\Phi = 0$.

The Navier–Stokes equations become:

$$\partial_t(\nabla\Phi) + \frac{1}{\rho}\nabla P \approx -g\hat{z} \qquad \text{(if velocities are small)}$$

$$\hookrightarrow \nabla\left[\partial_t\Phi + \frac{1}{\rho}P + gz\right] = 0 \to \partial_t\Phi + \frac{P}{\rho} + gz = \text{const.}$$

At interface: $P_0 - P(z=h+\zeta) = \alpha\nabla_\perp^2\zeta$

Combine: $\partial_t\Phi(z=h+\zeta) + \frac{1}{\rho}\left[\cancel{P_0} - \alpha\nabla_\perp^2\zeta\right] + g\zeta = \cancel{\text{const.}}$

Continuity of interface: $v_z(z=h+\zeta) \approx v_z(z=h) = \partial_t\zeta$

or in other words: $\partial_z \Phi(z=h) = \partial_t \zeta(z=h)$.

At $z=0$: $v_z = 0$ or $\partial_z \Phi(z=0) = 0$.

Because we have rotation symmetry in x & y and translation sym. we can Fourier transform in x, y, t:

Ansatz: $\zeta = \zeta_0 \cdot e^{ikx - i\omega t}$ and $\Phi = f(z)\, e^{ikx - i\omega t}$.

From $\nabla^2 \Phi = 0$: $(\partial_z^2 - k^2) f(z) = 0$

Since $f'(0) = 0$: $f(z) = f_0 \cdot \cosh(kz)$

Bdy cond at $z=h$: $-i\omega \zeta_0 = k f_0 \sinh(kh)$

Last equation: $-i\omega f_0 \cosh(kh) + \frac{\alpha}{\rho} k^2 \zeta_0 + g \zeta_0 = 0$

$$\left[\frac{(-i\omega)^2}{k \tanh(kh)} + g + \frac{\alpha}{\rho} k^2 \right] f_0 = 0$$

Dispersion relation: $\boxed{\omega = \pm \sqrt{\left(g + \frac{\alpha}{\rho} k^2\right) k \tanh(kh)}}$

"gravity waves"

For ocean waves (see HW4 and later discussion today) it is natural to neglect surface tension.

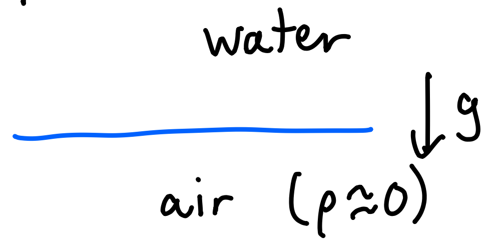
If $\alpha \to 0$: "deep water" waves $(kh \gg 1)$:

$$\omega = \pm \sqrt{gk}$$

"shallow water" waves $(kh \ll 1)$:

$$\underbrace{\omega = \pm \sqrt{gh}\; k}_{\text{linearly dispersing waves, like sound}}$$

Example 4: water droplets.



At this interface we should expect an instability b/c heavy water wants to fall through lighter air!

Take $g \to -g$ in above solution! $(h \to \infty)$

$$\omega = \pm \sqrt{\underbrace{\left(\frac{\alpha}{\rho}k^2 - g\right)}_{\text{instability when } k < k_c = \sqrt{\frac{\rho g}{\alpha}}} k}$$

For water-air interface: $\rho \sim 10^3 \ \mathrm{kg/m^3}$, $g \sim 10 \ \mathrm{\frac{m}{s^2}}$, $\alpha \sim 0.07 \ \frac{N}{m}$

$$k_c \sim 400 \ \mathrm{m^{-1}} \quad \text{or} \quad \underbrace{\lambda_c = \frac{2\pi}{k_c} \sim 1.5 \ \mathrm{cm}}_{\text{critical size of water droplet that will form!}}$$