PHYS 7810
Hydrodynamics
Spring 2026

Lecture 17
Atmospheric fluid flows

March 10

Today we will study Navier-Stokes in rotating reference frame.

Rotating vs. inertial frame:

$x_i(t) = R_{ij}(t)\,X_j(t)$ with e.g. $R_{ij}(t) = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\hookrightarrow \ddot{x} = R_{ij}\ddot{X}_j + 2\dot{R}_{ij}\dot{X}_j + \ddot{R}_{ij}X_j$$

$\downarrow$ define $\dot{R}_{ij} = R_{ik}\varepsilon_{klj}\Omega_l$

$$= R_{ij}\left[\ddot{X}_j + 2\varepsilon_{jlk}\Omega_l\dot{X}_k + \varepsilon_{jlk}\varepsilon_{kmn}\Omega_l\Omega_m X_n\right]$$

So in rotating frame we see:

effective forces: $\vec{f} = \underbrace{-2\vec{\Omega}\times\vec{v}}_{\text{Coriolis}} - \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{x})}_{\text{centrifugal}}$

$$= -\Omega^2\vec{x} + \vec{\Omega}(\vec{\Omega}\cdot\vec{x})$$

$$= -\frac{1}{2}\nabla\left[\Omega^2x^2 - (\vec{\Omega}\cdot\vec{x})^2\right]$$

Hence we find incompressible Navier-Stokes equations

$$\partial_t\vec{v} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla P' + \nu\nabla^2\vec{v} - 2\vec{\Omega}\times\vec{v}$$

with $P' = P - \frac{\rho}{2}\left[\Omega^2 x^2 - (\vec{\Omega}\cdot\vec{x})^2\right]$

We now identify 2 dimensionless parameters:

Reynolds: $\mathcal{R} \sim \frac{\vec{v}\cdot\nabla\vec{v}}{\nu\nabla^2\vec{v}} \rightarrow \frac{v_{typ}\, l_{typ}}{\nu}$

Rossby: $\Upsilon \sim \frac{\vec{v}\cdot\nabla\vec{v}}{\Omega\times\vec{v}} \rightarrow \frac{v_{typ}}{l_{typ}\,\Omega}$

For atmospheric flow (e.g. storm):

$v_{typ} \sim 10\ m/s \qquad l_{typ} \sim 1000\ km \qquad \nu \sim 10^{-5}\ \frac{m^2}{s}$

$\Omega \sim \frac{2\pi}{1\ day} \sim \frac{2\pi}{24\cdot 60^2}\ s^{-1} \sim 10^{-4}\ s^{-1}$

$\Rightarrow \mathcal{R} \sim 10^{13} \qquad \Upsilon \sim 10^{-1}$ so Coriolis is very important!

What do Coriolis-dominated flows look like?

geostrophic flow = $\Upsilon \ll 1$, static $(\partial_t = 0)$:

$$0 = -\frac{1}{\rho}\nabla P' + \cancel{\nu\nabla^2\vec{v}} - 2\vec{\Omega}\times\vec{v}$$

ignore up to boundary layer...

$\downarrow$

solve for $\vec{v}$

In our atmospheric flow we could estimate that

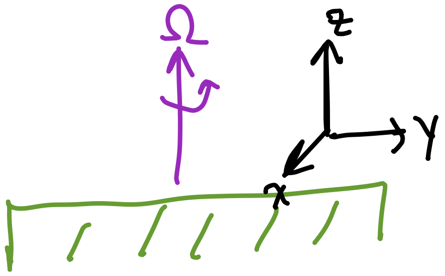
$\frac{\Delta P'}{l_{typ}} \sim 2\rho\Omega v_{typ}$, $\rho \sim 1\ \frac{kg}{m^3} \Rightarrow \Delta P' \sim 2\ kPa \sim 0.02\ atm$

So the typical fluctuations in atmospheric pressure are on the order of a few percent.

If $\nabla P' = -2\rho \vec{\Omega} \times \vec{v}$ then $\nabla \times \nabla P' = 0$

$0 = \nabla \times (\vec{\Omega} \times \vec{v})$ or $\boxed{0 = (\vec{\Omega} \cdot \nabla)\vec{v}}$ Taylor-Proudman Theorem

This means that velocity only varies perpendicular to rotation...

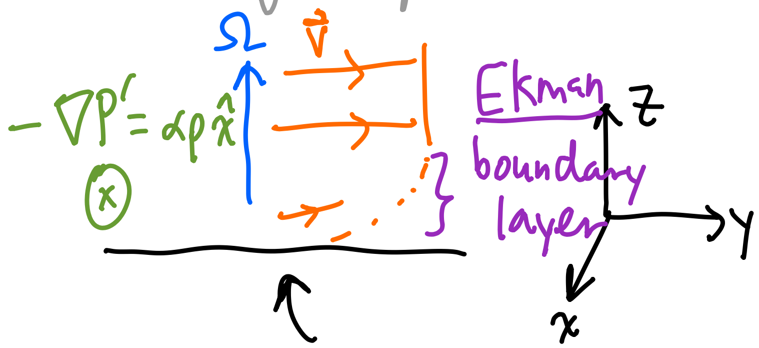


$\Omega \, \partial_z \vec{v} = 0$

Since $v_z(z = \text{ground}) = 0 \ldots$ $v_z = 0$ and

(v_x, v_y) only depend on (x, y) $\rightarrow$ $\underbrace{\text{flow becomes 2d}}_{\text{conclusion holds as long as } Y \ll 1}$

We might expect that, as in lec. 13, viscous effects become important in geostrophic flow in a boundary layer...



need to impose no-slip at $z = 0$.

Far from boundary: $-\alpha \hat{x} = 2\Omega \hat{z} \times \vec{v}$ or $\vec{v} = \frac{\alpha}{2\Omega} \hat{y}$

Claim: $\nabla P \approx$ const. in boundary layer

(similar to before but today we won't explicitly verify...)

If $Y \ll 1$: $0 \approx \nu \nabla^2 \vec{v} - 2\vec{\Omega} \times \vec{v} - \frac{\nabla P'}{\rho}$

Write: $\vec{v} = v_0 \hat{y} + \delta v_y \hat{y} + \delta v_x \hat{x}$

Boundary conditions: $\delta v_x(z=0)=\delta v_x(z=\infty)=0$

$\delta v_y(z=\infty)=0 \qquad \delta v_y(z=0)=-v_0.$

The v_0 term cancels the constant pressure gradient so we just get:

$$\nu\partial_z^2\delta v_y=2\Omega\delta v_x \qquad \text{and} \qquad \nu\partial_z^2\delta v_x=-2\Omega\delta v_y$$

Write $u=\delta v_y+i\delta v_x$: $\quad \nu\partial_z^2 u=-i\cdot 2\Omega u$

So $u(z)=Ae^{\sqrt{-2i\Omega/\nu}\,z}+Be^{-\sqrt{-2i\Omega/\nu}\,z}$

$\sqrt{-i}=e^{-i\pi/4}=\frac{1-i}{\sqrt{2}}$ so $A=0$ ($z=\infty$ bdy cond.)

$u(0)=-v_0$ fixes $B=-v_0$.

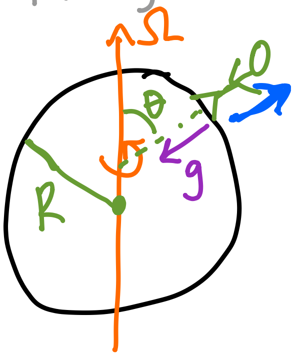
Hence:

$$\left.\begin{aligned} v_y(z)&=v_0\left(1-e^{-\sqrt{\Omega/\nu}\,z}\cos\left(\sqrt{\frac{\Omega}{\nu}}z\right)\right)\\ v_x(z)&=-v_0e^{-\sqrt{\Omega/\nu}\,z}\sin\left(\sqrt{\frac{\Omega}{\nu}}z\right)\end{aligned}\right\}$$

Ekman boundary layer has thickness $\delta_E=\sqrt{\frac{\nu}{\Omega}}$

On HW5 you will explore how Ekman boundary layers cause rapid mixing in rotating fluids (e.g. stirring coffee).

Now let's talk about fluid dynamics on the surface of the rotating Earth a bit more concretely:



Claim: $\Omega_{\text{eff}}=\Omega\cos\theta$

↳ hydrodynamics of atmosphere/ocean $\approx$ 2d!

First observe that:

$$\frac{\text{gravity}}{\text{Coriolis}}\sim\frac{g}{v_{typ}\Omega}\sim\frac{10}{10\cdot10^{-4}}\sim10^4\gg1$$

This means that we need to first account for gravity when solving fluid mechanics equations:

$$\partial_t \vec{v} \approx 0 = -\frac{1}{\rho}\nabla p' - g\hat{r} = -\frac{1}{\rho}\nabla\underbrace{(P' - \rho g r)}_{\approx\ \text{constant.}}$$

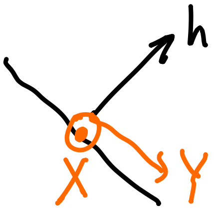
Or: $$\underbrace{P - \frac{\rho\Omega^2}{2}R^2\sin^2\theta + \rho g r}= \text{const.}$$

effective atmospheric height: $h = r - R - \frac{\Omega^2}{2g}R^2\sin^2\theta$

Now add back in Coriolis as a small perturbation:

$$\partial_t \vec{v} \approx 0 = -\frac{1}{\rho}\nabla(P + \rho g h) - 2\vec{\Omega}\times\vec{v}$$

Introduce a new coordinate system:



(local rectangular coords)

Atmosphere is thin so:

$$0 = \nabla\cdot\vec{v} = \frac{\partial v_X}{\partial X} + \frac{\partial v_Y}{\partial Y} + \frac{\partial v_h}{\partial h}$$

$\frac{\partial}{\partial h} \gg \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}$ so $\boxed{v_h \ll v_X, v_Y.}$

Write $\vec{\Omega} = \Omega(\cos\theta\,\hat{h} - \sin\theta\,\hat{Y})$:

$$\vec{\Omega}\times\vec{v} = \underbrace{\Omega\cos\theta}_{\Omega_{\text{eff}}}\left[v_X\hat{Y} - v_Y\hat{X}\right] - \underset{v_h \ll v_X,\, v_Y}{\cancel{\Omega\sin\theta\, v_h\hat{X}}} + \underset{\text{small correction to hydrostatics/ gravity}}{\cancel{\Omega\sin\theta\, v_X\hat{h}}}$$

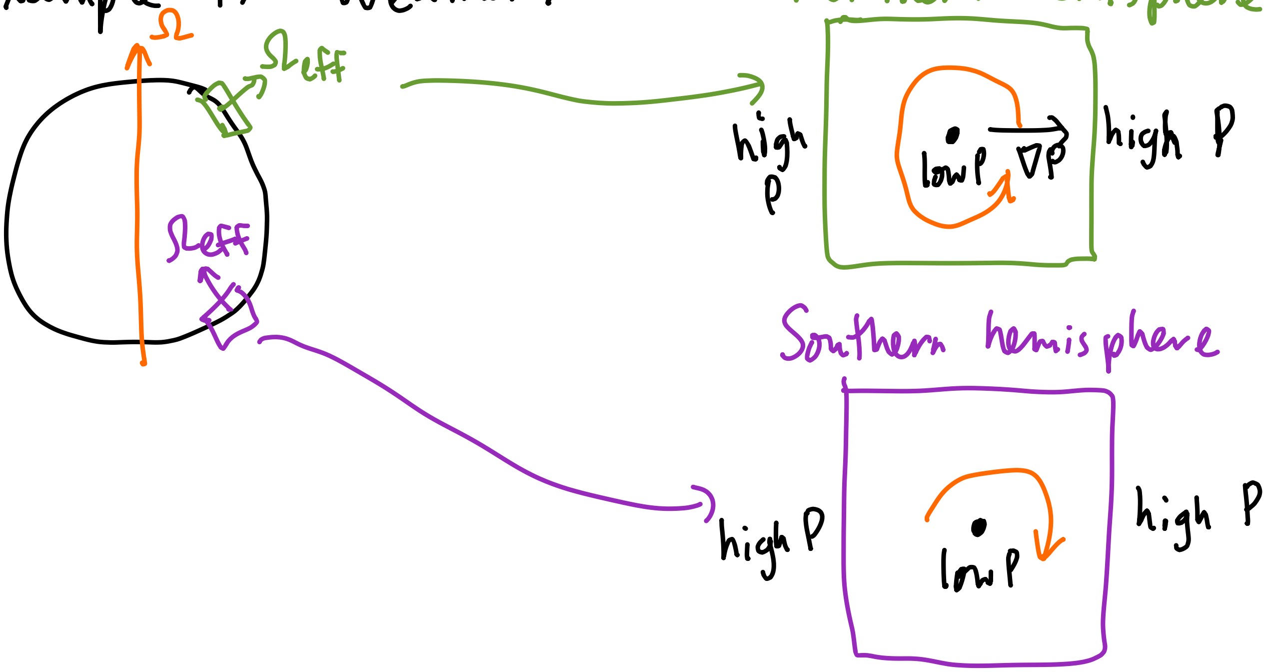
Flow is approximately 2d along the surface of Earth!

Conclude: if $\vec{v} = v_x \hat{x} + v_y \hat{y}$:

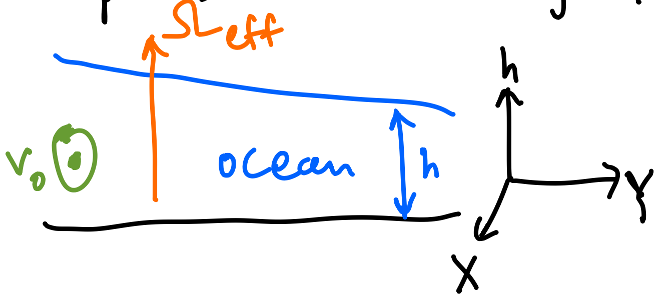
$$\nabla \cdot \vec{v} \approx 0$$

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \frac{1}{\rho}\nabla \tilde{p} - \nu \nabla^2 \vec{v} \approx -2\Omega_{\text{eff}}\,\hat{z} \times \vec{v}$$

Example 1: Weather:



Example 2: ocean height?



If pressure similar at ocean surface then:

$$-2\Omega_{\text{eff}} v_0 \hat{y} = \frac{1}{\rho}\nabla \tilde{p} \approx g \nabla h$$

$$\partial_y h \sim -\frac{2\Omega_{\text{eff}} v_0}{g}$$

ocean currents at $v_0 \sim 0.1$ m/s $\Rightarrow$ $\partial_y h \sim 10^{-6}$

or $\Delta h \sim 1$ m over 1000 km

These effects are observable in data.