PHYS 7810
Hydrodynamics
Spring 2026

Lecture 18
Convection

March 12

Today we will discuss thermal transport, especially in fluids in motion. Let's begin by reviewing the full hydrodynamic equations:

$$\partial_t \rho + \partial_i(\rho v_i) = 0$$

$$\partial_t(\rho v_i) + \partial_j\left[P\delta_{ij} + \rho v_i v_j - \tilde{\tau}_{ji}\right] = 0 \qquad \text{where}$$

$$\tilde{\tau}_{ji} = \zeta\,\partial_k v_k \delta_{ji} + \eta\left(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}\delta_{ij}\partial_k v_k\right)$$

$$\partial_t \varepsilon + \partial_i\left[(\varepsilon+P)v_i - \kappa\partial_i T - v_j\tilde{\tau}_{ji}\right] = 0$$

Today we can no longer neglect energy.
It is useful to change variables and write:

$$\underset{\substack{\uparrow\\ \text{energy}\\ \text{density}}}{\varepsilon} = \underbrace{\tfrac{1}{2}\rho v^2}_{\text{kinetic}} + \underbrace{\rho\hat{\varepsilon}}_{\text{internal}} \qquad \underset{\substack{\uparrow\\ \text{entropy density.}}}{s} = \rho\hat{s}$$

The idea is that for many liquids & gases, temperature will most strongly affect $\hat{\varepsilon}$ (and $\hat{s}$) rather than ρ. So this change of variable will eventually lead to simpler equations.

Quick review of thermodynamics from lec 7 & 8.

$$d\varepsilon = T\,ds + \tilde{\mu}\,d\rho + v_i\,dg_i \quad \text{and} \quad \varepsilon + P = \rho\left(\tilde{\mu} + \frac{v^2}{2}\right) + Ts$$

$$\hookrightarrow \; d\left(\rho\hat{\varepsilon} + \tfrac{1}{2}\rho v_i v_i\right) = T\,d(\rho\hat{s}) + \tilde{\mu}\,d\rho + v_i\,d(\rho v_i)$$

$$\rho\,d\hat{\varepsilon} = \rho T\,d\hat{s} + \left(\tilde{\mu} + \frac{v^2}{2} + T\hat{s} - \hat{\varepsilon}\right)d\rho$$

or: $$d\hat{\varepsilon} = T\,d\hat{s} + \frac{P}{\rho^2}\,d\rho$$

So: $$\partial_t \varepsilon = \left(\hat{\varepsilon} + \frac{v^2}{2}\right)\partial_t\rho + \rho v_i\,\partial_t v_i + \rho T\,\partial_t\hat{s} + \frac{P}{\rho}\partial_t\rho$$

$$= \frac{\varepsilon + P}{\rho}\partial_t\rho + \rho v_i\,\partial_t v_i + \rho T\,\partial_t\hat{s}$$

$$= -\partial_i\left[\frac{\varepsilon+P}{\rho}\rho v_i - v_j\tilde{\tau}_{ji} - \kappa\,\partial_i T\right]$$

or: $$\rho T\,\partial_t\hat{s} = -\rho v_i\,\partial_i\left(\frac{\varepsilon+P}{\rho}\right) - \rho v_i\left(-v_j\partial_j v_i - \frac{\partial_i P}{\rho} + \frac{\partial_j\tilde{\tau}_{ji}}{\rho}\right) + \partial_i\left(v_j\tilde{\tau}_{ji} + \kappa\,\partial_i T\right)$$

$$= -\rho v_i\,\partial_i\hat{\varepsilon} + (\partial_i v_j)\tilde{\tau}_{ji} + \partial_i(\kappa\,\partial_i T) + \frac{P v_i}{\rho}\partial_i\rho$$

or: $$\boxed{\rho T\left(\partial_t\hat{s} + v_i\,\partial_i\hat{s}\right) = \partial_i v_j\,\tilde{\tau}_{ji} + \partial_i(\kappa\,\partial_i T)}$$

heat transfer equation

Convection: transfer of heat by moving fluid

Today we will study a few paradigms for convection in ordinary fluids!

In most fluids, we can approximate ρ varies slowly so:

Write: $\hat{s}(P, T)$ and approximate: $d\hat{s} \approx \underbrace{\left.\frac{\partial \hat{s}}{\partial T}\right|_P}_{c_p/T} dT$

pressure (P), temperature (T)

approx κ = const.

So: $\rho c_p (\partial_t T + v_i \partial_i T) = \partial_i v_j \tilde{\tau}_{ji} + \kappa \partial_i \partial_i T$

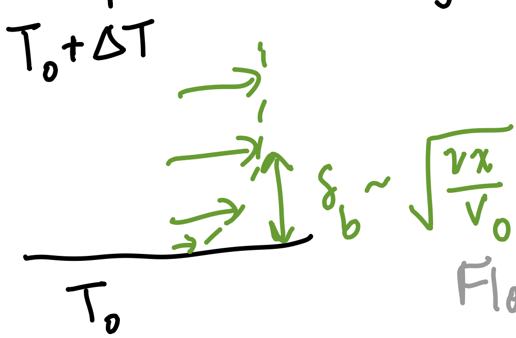
As before it will be helpful to identify some dimensionless parameters that reveal the dominant effects. Consider

$$\frac{\eta v_{typ}^2 \ell_{typ}^{-2}}{\kappa \Delta T \ell_{typ}^{-2}} = \frac{\eta v_{typ}^2}{\kappa \Delta T} = B \quad \text{(Brinkman number)}$$

Today: $B \ll 1$ (neglect heating due to viscosity). Then:

$$\partial_t T + \vec{v} \cdot \nabla T = \chi \nabla^2 T \quad \text{where} \quad \chi = \frac{\kappa}{\rho c_p} = \text{thermal diffusion constant}$$

Example: heating in boundary layer:



$$v_x \partial_x T + v_y \partial_y T = \chi \nabla^2 T \approx \chi \partial_y^2 T$$

Flow depends on another dimensionless parameter:

Prandtl number $\mathcal{P} = \frac{\nu}{\chi}$

① Assume $\mathcal{P} \ll 1$. $T \approx T_0$ inside viscous boundary layer.

$$v_0 \partial_x T = \chi \partial_y^2 T$$

So T varies over $\Delta y \sim \delta_h$: $\frac{\chi}{\delta_h^2} \sim \frac{v_0}{x}$ or $\frac{\mathcal{P}}{\delta_h^2} \sim \frac{1}{\delta_b^2}$

So $\delta_h \sim \delta_b \mathcal{P}^{-1/2}$.

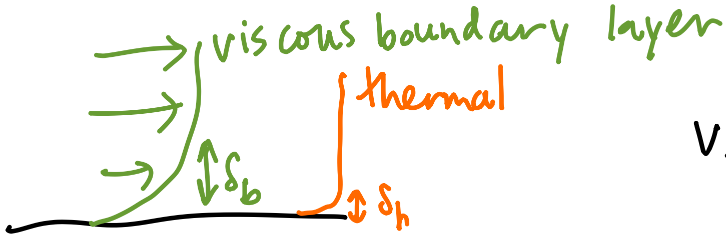
The typical heat flux at the surface is given by:

$$\mathcal{E}_y = -\kappa \partial_y T \sim -\frac{\kappa \Delta T}{\delta_h} \sim -\kappa \Delta T \sqrt{\frac{v_0}{\chi \ell}}$$

ℓ is plate size.

Notice that the speed of heat transfer is sped up by fluid flow!

(2) Assume $P \gg 1$. Now: we will show that:



$$v_x \partial_x T + v_y \partial_y T = \chi \partial_y^2 T$$

$$v_x(y=\delta_h)\frac{\Delta T}{x} \sim v_0 \frac{\delta_h}{\delta_b}\frac{\Delta T}{x} \qquad \frac{\chi \Delta T}{\delta_h^2}$$

So: $\delta_h^3 \sim \dfrac{\chi x \delta_b}{v_0} \sim P^{-1}\delta_b^3$

Heat flux: $\mathcal{E}_y = -\kappa \partial_y T \sim -\dfrac{\kappa \Delta T}{\delta_h} = -\dfrac{\kappa \Delta T}{\delta_b} P^{1/3}$

$$\sim -\kappa \Delta T \sqrt{\frac{v_0}{\nu^{1/3}\chi^{2/3}\ell}}$$

When χ is small the viscous boundary layer smears out the thermal boundary layer and slows down the rate of heating.

But often, convection arises when a strong temperature gradient itself begins to drive fluid flow. To understand how this can happen let's first think about heating a fluid:

$$\left.\frac{\partial \rho}{\partial T}\right|_p = -\beta \rho$$

coefficient of thermal expansion: $\beta > 0$.

Assume $\beta \Delta T \ll 1$ so $\Delta\rho/\rho_0 \ll 1$

So we approximately maintain incompressibility. Moreover we want ρ to depend only on T:

$$\left.\frac{\partial\rho}{\partial P}\right|_T \sim \frac{1}{v_s^2}, \quad \text{so:} \quad \frac{\Delta P}{v_s^2} \ll \beta\rho\Delta T$$

e.g. hydrostatic pressure: $\dfrac{g\Delta z}{v_s^2} \ll \beta\Delta T$ (OK if v_s large...)

So then our equations of convection are approximately:

$$\boxed{\nabla\cdot\vec{v}=0}$$

Boussinesq approximation (incompressible flow)

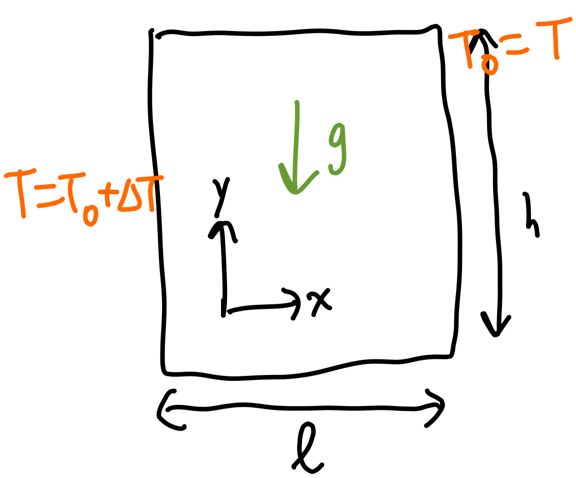
$$\boxed{\partial_t\vec{v}+\vec{v}\cdot\nabla\vec{v}+\frac{1}{\rho(T)}\nabla P-\nu\nabla^2\vec{v}=\vec{g}}$$

← gravitational acceleration/ other external forces.

where $\rho(T)\approx\rho_0(1-\beta(T-T_0))$

$$\boxed{\partial_t T+\vec{v}\cdot\nabla T=\chi\nabla^2 T}$$

To look at a minimal model for convection, consider the following:



Assume ΔT small, flow velocity small. If $\partial_t=0$:

$v_x=\partial_y\psi$ and $v_y=-\partial_x\psi$

$\nabla^2 T=0$

$$\frac{1}{\rho(T)}\partial_x P=\nu\nabla^2\partial_y\psi$$

$$\frac{1}{\rho(T)}\partial_y P=-\nu\nabla^2\partial_y x-g$$

Solve for T: $T = \Delta T \cdot \left(1 - \frac{x}{\ell}\right) + T_0$

Neglecting higher-order $\Delta T \cdot \vec{v}$ terms...

$$\partial_y \partial_x P - \partial_x \partial_y P \approx \rho_0 \nu \nabla^2 \nabla^2 \psi + \underbrace{\partial_x \left[g \rho(T) \right]}_{= -\partial_x T \cdot g \beta \rho_0 = + g \beta \rho_0 \frac{\Delta T}{\ell}}$$

$$\nabla^2 \nabla^2 \psi \approx \frac{g \beta \Delta T}{\nu \ell}$$

↳ Write $\psi = \psi_h + \psi_p$ where $\nabla^2 \nabla^2 \psi_h = 0$ and:

$$\psi_p = \frac{1}{64} \frac{g \beta \Delta T}{\nu \ell} (x^2 + y^2)^2$$ is a particular solution.

Now choose ψ_h so: $\psi_h + \psi_p = 0$ and $\partial_n (\psi_h + \psi_p) = 0$ at walls of box. The exact solution is not so enlightening but could be found using standard methods.

But we can get a heuristic sense of the solution:

Assuming $h \sim \ell$:

$$v_{typ} \sim \frac{\psi}{\ell} \sim \underline{\frac{g \beta \Delta T \ell^2}{\nu}}$$

ideal gas law: $\rho T = P$

For ideal gas like air: $\beta = -\frac{\partial \rho}{\partial T}\Big|_P = \frac{1}{T}$

so w/ 1% relative temperature difference:

$$v_{typ} \sim 0.01 \cdot \frac{10}{10^{-5}} \left(\frac{\ell}{1\,m}\right)^2 \frac{m}{s} \sim \left(\frac{\ell}{1\,cm}\right)^2 m/s$$

This estimate is a bit too large for human scale flow!

Since this flow will actually be fast, a better estimate is to neglect viscosity and instead keep track of inertia:

$$\frac{1}{\ell}\,\frac{v_{typ}^2}{\ell} \sim \frac{g\beta\Delta T}{\ell} \quad \text{or} \quad v_{typ} \sim \sqrt{\beta\Delta T \cdot \ell g}$$

Estimate for kitchen oven w/ $\beta\Delta T = 0.01$: $\ell \sim 1\,m$ and $v_{typ} \sim 0.3\ m/s$

which is much more realistic