PHYS 7810
Hydrodynamics
Spring 2026

Lecture 19
Fluids with modified spacetime symmetries

March 24

Today we will build hydro EFTs for fluids w/ reduced spacetime symmetries. For simplicity we focus only on conserved mass & momentum.

$$\mathcal{L}_{MSR} = \pi \partial_t \rho + \pi_i \partial_t g_i \underbrace{- \partial_i \pi J_i - \partial_j \pi_i \tau_{ji}}_{\text{ensures that } \int \rho,\ \int g_i \text{ are conserved}}$$

What are constitutive relations J_i & τ_{ji}?

Assume: generalized time-reversal symmetry: $\mu = \frac{\delta \Phi}{\delta \rho}$, $v_i = \frac{\delta \Phi}{\delta g_i}$

$$t \to -t \qquad x_i \to A_{ij} x_j \qquad \rho \to \rho \qquad g_i \to -A_{ij} g_j$$

$$\partial_i \to A_{ji} \partial_j \quad \text{and} \quad A_{ij} A_{jk} = \delta_{ik} \qquad \pi \to -\pi + i\mu \qquad \pi_i \to A_{ji}(\pi_j - i v_j)$$

Neglecting dissipative corrections, what J_i & τ_{ij} possible?

$$\mathcal{L}_{MSR} \to (\pi - i\mu)\partial_t \rho + A_{ji}(\pi_j - i v_j)\partial_t (A_{ik} g_k) - (A_{ji}\partial_j)(-\pi + i\mu)(\mathbb{T}\cdot J)_i$$
$$- (A_{\ell j}\partial_\ell)(A_{ki}(\pi_k - i v_k))(\mathbb{T}\cdot \tau)_{ji}$$

$$\stackrel{?}{=} \mathcal{L}_{MSR} - i\left[\partial_t \Phi - \partial_i \underbrace{S_i}_{\text{emergent entropy current?}}\right] \ ?$$

Note: $A_{ji}(\pi_j - iv_j)\partial_t(A_{ik}g_k) = \delta_{jk}(\pi_j - iv_j)\partial_t g_k = (\pi_i - iv_i)\partial_t g_i$ ✓

For this to work out we also need:

$$(T\cdot J)_i = -A_{ij}J_j \quad \text{and} \quad (T\cdot \tau)_{ji} = A_{jm}A_{in}\tau_{mn}$$

and: $$\underline{\partial_i\mu\, J_i} + \underline{\partial_i v_j\, \tau_{ij}} = \partial_i \mathcal{S}_i$$
$$= \underline{\frac{\partial \mathcal{S}_i}{\partial \mu}\partial_i\mu} + \underline{\frac{\partial \mathcal{S}_i}{\partial v_j}\partial_i v_j}$$

$$J_i = \frac{\partial \mathcal{S}_i}{\partial \mu} \qquad \tau_{ij} = \frac{\partial \mathcal{S}_i}{\partial v_j} \qquad \text{in ideal hydro!}$$

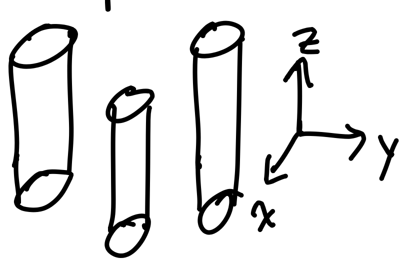
Sanity check: neglecting energy, Galilean fluid has

$$\mathcal{S}_i = P\left(\mu + \tfrac{v^2}{2}\right)v_i : \qquad J_i = \frac{\partial P}{\partial \mu}v_i = \rho v_i$$
$$\tau_{ij} = P\delta_{ij} + v_j\frac{\partial P}{\partial(v^2/2)}v_i = P\delta_{ij} + \rho v_i v_j$$

This time we will proceed more efficiently by writing down all possible terms in $\mathcal{S}_i$ and deducing J_i and τ_{ij} from there!

Calculating dissipative terms will proceed similarly to before: just need to write down $\pi(\pi - i\mu)$ type motifs for consistency w/ T...

Example 1: 3d fluid of cylindrical particles...



- time-reversal symmetry
- particles aligned w/ z-axis.

So spacetime symmetries are:

① $z \to -z$

② $y \to -y$

③ $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

To build S_i we should look for invariant tensors under ①-③:

Invariants: $\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ AND $\lambda_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Note: $n_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is NOT invariant under ①

Under time-reversal: $\frac{\partial S_i}{\partial \mu} = J_i \to -J_i$, so $S_i \to -S_i$ under T

Since $\mu \to \mu$ and $v_i \to -v_i$ under T, most general

$$S_i = f(\mu, v_i v_i, \underbrace{v_i \lambda_{ij} v_j}_{= v_z^2}) v_i + \underbrace{h(\mu, v_i v_i, v_z^2)}_{\text{T-even invariant building blocks}} \underbrace{\lambda_{ij} v_j}_{\text{another T-odd motif allowed}}$$

So: $J_i = \frac{\partial f}{\partial \mu} v_i + \frac{\partial h}{\partial \mu} \lambda_{ij} v_j$

$$\tau_{ij} = f\delta_{ij} + h\lambda_{ij} + \frac{\partial f}{\partial \frac{v^2}{2}} v_i v_j + \frac{\partial f}{\partial \frac{v_z^2}{2}} v_i \lambda_{kj} v_k + \frac{\partial h}{\partial \frac{v^2}{2}} \lambda_{ik} v_k v_j + \frac{\partial h}{\partial v_z^2/2} \lambda_{ik} \lambda_{jl} v_k v_l$$

Note: $\tau_{zx} \neq \tau_{xz}$ because no rotational symmetry in xz-plane!

Dissipative corrections? Let's focus on the terms that are present in linear response regime...

$$\mathcal{L}_{MSR,\,diss} = i\partial_i \pi_j \eta_{ijkl} \partial_k(\pi_l - iv_l) + i\partial_i \pi\, \sigma_{ij}\, \partial_j(\pi - i\mu)$$

We cannot mix μ & v terms due to time-reversal symmetry.

Need to build η_{ijkl} & σ_{ij} out of invariants δ_{ij} & λ_{ij}:

$$\sigma_{ij} = \sigma_1 \delta_{ij} + \sigma_2 \lambda_{ij}$$

By T, $\eta_{ijkl} = \eta_{klij}$ is required...

$$\eta_{ijkl} = \zeta\delta_{ij}\delta_{kl} + \zeta_1(\delta_{ij}\lambda_{kl} + \delta_{kl}\lambda_{ij}) + \zeta_2\lambda_{ij}\lambda_{kl} \rightarrow = \lambda_{ik}\lambda_{jl}\cdots$$
$$+ \eta_1\delta_{ik}\delta_{jl} + \eta_2\delta_{il}\delta_{jk} + \eta_3\delta_{ik}\lambda_{jl} + \eta_4\lambda_{ik}\delta_{jl}$$
$$+ \eta_5(\delta_{il}\lambda_{jk} + \delta_{jk}\lambda_{il})$$

xy-continuous rotational symmetry: $\eta_2 = \eta_1$ and $\eta_4 = \eta_3$

$\eta_{ij,kl}$ must be positive semidefinite:

e.g. $\zeta \geq 0$, $\zeta_2 \geq 0$, $\zeta_1^2 \leq \zeta\zeta_2$ ...

Proliferation of dissipative coefficients is similar to the large number of elastic moduli in anisotropic solids...

Example 2: 2d fluid w/ PT symmetry:

Use generalized T w/ $A_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Assume rotational symmetry.

↳ Claim: invariants are δ_{ij}.

Levi-Civita

and under T, $\varepsilon_{ij} \rightarrow -\varepsilon_{ij}$

Let's again start w/ ideal fluid.

Try: $S_i = f(\mu, v^2/2)v_i + \cancel{h(\mu, v^2/2)\varepsilon_{ij}v_j}$?

$\hookrightarrow J_x = \cdots + \underbrace{\frac{\partial h}{\partial \mu} v_y}$

$\rightarrow \frac{\partial h}{\partial \mu} v_y$ under T while $J_x \rightarrow -J_x$

So S_i is same as ordinary fluid and:

$J_i = \rho v_i$ where $\rho = \frac{\partial f}{\partial \mu}$; $T_{ji} = \frac{\partial f}{\partial v^2/2} v_i v_j + \delta_{ij} f$ ← pressure.

In this theory the derivative corrections are more interesting.

Again let's restrict to terms contributing to linear EOMs.

$$\mathcal{L}_{MSR,diss} = i\partial_i\pi\,\sigma_{ij}\,\partial_j(\pi - i\mu) + i\partial_i\pi_j\,\eta_{ijkl}\,\partial_k(\pi_l - iv_l)$$

"Hall conductivity" (incoherent)

Suppose $\sigma = \sigma_0\delta_{ij} + \sigma_H\varepsilon_{ij}$

Under T: $\rightarrow i\,\partial_i(-\pi + i\mu)(\sigma_0\delta_{ij} - \sigma_H\varepsilon_{ij})\partial_j(-\pi)$

$$= i\sigma_0\,\partial_i(\pi - i\mu)\,\partial_i\pi - \underbrace{i\sigma_H\,\partial_i(\pi - i\mu)\varepsilon_{ij}\partial_j\pi}_{= +i\sigma_H\,\partial_j\pi\,\varepsilon_{ji}\,\partial_i(\pi - i\mu)}$$

So both $\sigma_0, \sigma_H \neq 0$ are allowed.

Positivity constraints: $\sigma_0 \geq 0$ but σ_H unconstrained b/c:

$i\sigma_H\,\partial_i\pi\,\varepsilon_{ij}\,\partial_j\pi = 0 \rightarrow$ no noise variance $\rightarrow$ dissipationless!

Let's now try the same thing w/ η_{ijkl}:

Hall viscosity

$$\eta_{ijkl} = \zeta\,\delta_{ij}\delta_{kl} + \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\right) + \eta_H\left(\delta_{ik}\varepsilon_{jl} + \delta_{il}\varepsilon_{jk}\right)\ ?$$

Don't write $\varepsilon_{ij}\varepsilon_{kl}$ b/c the stress tensor needs to be symmetric due to angular momentum conservation!

Note: $\delta_{ik}\varepsilon_{jl} + \delta_{il}\varepsilon_{jk} = \sigma^x_{ji}\sigma^z_{kl} - \sigma^z_{ji}\sigma^x_{kl}$ so $\eta_{ijkl} = \eta_{jikl} = \eta_{ijlk}$

So under T: $\eta_H\,\partial_i\pi_j\left(\sigma^x_{ij}\sigma^z_{kl} - \sigma^z_{ij}\sigma^x_{kl}\right)\partial_k(\pi_l - iv_l)$

$$\rightarrow \eta_H\,\partial_i(\pi_j - iv_j)\left(-\sigma^x_{ij}\sigma^z_{kl} + \sigma^z_{ij}\sigma^x_{kl}\right)\partial_k\pi_l \quad \text{invariant!}$$

So we can also include Hall viscosity in the PT-symmetric fluid!

Hall viscosity can arise in

- electron liquids in metals where parity broken
- active chiral matter

etc.

What does Hall viscosity do in the Navier–Stokes equations?

Notice: If $\tau_{ij} = \tau_{ij}^{\text{usual}} - \eta_H(\delta_{ik}\varepsilon_{jl} + \delta_{il}\varepsilon_{jk})\partial_k v_l$ then:

if $\partial_i v_i = 0$, then:

$$\partial_j \tau_{ij} \to \partial_j \tau_{ij} - \underbrace{\eta_H \partial_i \partial_j \varepsilon_{jl} v_l}_{\eta_H \partial_i \omega}$$

where vorticity $\omega = \partial_j \varepsilon_{jl} v_l$

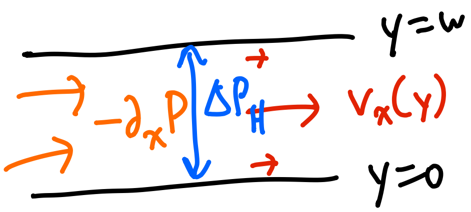
For incompressible flow (assuming Galilean-invariant thermodynamics)

$$\partial_t v_i + v_j \partial_j v_i + \frac{1}{\rho}\partial_i \underbrace{(P - \eta_H \omega)} - \frac{\eta}{\rho}\partial_j \partial_j v_i = 0$$

if η_H is constant

Hall viscosity only leads to an offset in pressure!

As an illustration of this let's consider Poiseuille flow:



From lecture 12:

$$v_x(y) = -\frac{\partial_x P}{2\eta} y(w - y)$$

y-momentum equation: $\partial_y(P - \eta_H \omega) = 0$ or $P = \eta_H \omega$

$$\omega = -\partial_y v_x = \frac{\partial_x P}{2\eta}(w - 2y)$$

So we induce a change in pressure in the vertical direction!

$$\Delta P_H = P(y=w) - P(y=0) = -\frac{\eta_H}{\eta}\partial_x P \cdot w$$